KÜPPERS - LORTZ INSTABILITY IN RAYLEIGH BENARD SYSTEM

by

SUPREETI DAS

DEPARTMENT OF PHYSICS

INDIAN INSTITUTE OF TECHNOLOGY KANPUR

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SUPREETI DAS

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TO

ANJANEYA

CERTIFICATE

This is to certify that the work contained in the thesis entitled "*Küppers Lortz instability in Rayleigh Benard system*" by *Supreeti Das*, has been carried out under my supervision. No part of this work has been submitted elsewhere for a degree.

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1. DR. J. K. BHATTACHARJEE

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This dissertation centres around a study of the Küppers-Lortz instability in fluids in Rayleigh Benard geometry. When a layer of fluid confined between two infinite horizontal surfaces is rotated about a vertical axis, Küppers and Lortz have determined that the stable steady state convection is in the form of two dimensional rolls. Above a critical rotation speed (characterized by the non-dimensional Taylor number) these steady state rolls get destabilized due to the growth of another set of rolls with a different orientation. This instability occurs immediately above the convection threshold. For a high Prandtl number fluid, since the destabilization cannot occur via a Hopf-bifurcation, there are no stable fixed points or limit cycles. Consequently the system shows irregular behaviour and is of great interest as a prototype for a system which shows transition

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to turbulence (weak) immediately above the convection threshold.

Since the hydrodynamic equations involve coupled non-linear partial differential equations which are difficult to solve, a Lorenz-like truncation of the actual hydrodynamic equations is carried out which results in a system of coupled non-linear ordinary differential equations. This system provides an adequate description of convection near the threshold and facilitates a detailed study of the various convection states near the onset.

In the present work, velocity, vorticity and the temperature fluctuations have been represented by Fourier modes upto second order. The critical Taylor number in the limit of very large Prandtl number, has been determined to be $23{\cdot}45\ \pi^4$ by rolls at an angle of 58° with respect to the original set. This is in complete agreement with the results of Küppers and Lortz.

If the layer of fluid is assumed to be of finite Prandtl number, the non-linear advective term in the hydrodynamic equation cannot be neglected. Our truncated system makes it possible to arrive at closed form expressions even at low Prandtl numbers. The threshold Taylor number is found to be considerably reduced. This is in agreement with the numerical works of Küppers and the experimental works of Niemella and Donnelly.

We then extend our study to consider the Küppers Lortz instability for an electrically conducting fluid which is placed in an external magnetic field. The joint effect of rotation and magnetic field on the onset of convection is particularly

Küppers-Lortz instability in this case shows the growth of a set of rolls at right angles to the original set at very low Taylor number indicating presumably the formation of square rolls rather than cylindrical ones.

CHAPTER I

INTRODUCTION

Instability in rotating fluids has drawn a lot of attention ever since the experimental evidence[1] of the onset of time dependent motion immediately above the threshold for convection was reported. Küppers and Lortz[2] and later Busse and Clever[3] have analyzed that this direct transition to the chaotic state is related to the onset of an instability which is characteristic of rotating fluids. In this dissertation we investigate the Küppers Lortz instability in a rotating Rayleigh Benard system using truncated models.

Section I of this chapter reviews the concepts related to Küppers Lortz instability and turbulence while Section II is a brief description of the work presented in the thesis.

SECTION I

REVIEW

1.1 Turbulence

A class of phenomena which is extremely sensitive to the variations in the initial conditions and is characterized by randomness and irregularities , is termed as turbulence[4]. For fluids in particular[5],turbulence is related to a time dependent complex flow in contrast to a laminar flow in which the fluid

travels along a well defined path with a velocity which is spatially and temporally uniform. Beyond a critical Reynolds number the laminar flow becomes turbulent. After turbulence sets in, two flows which are identical with slightly different initial conditions may ultimately evolve into two absolutely different flows. Motion which is initially periodic may become quasi-periodic or aperiodic and finally acquire a highly random character.

An extremely simplified explanation of turbulence is the absence of a unique solution of the equations of motion for asymptotically large times. When a system has stable solutions, turbulence does not set in as the system settles into one of these stable states eventually. However, if all the solutions are unstable, the system is forced to migrate in phase space spanned by these solutions without settling into any of them. Thus a conceptual model of turbulence[6] is a manifold of stationary solutions all of which are unstable to some other solution in the manifold such that the system constantly moves from the neighbourhood of one solution to another but never manages to settle into any of them. Therefore the phenomenon appears to be completely dependent on the qualitative features of the differential equations governing the system. The basic observation is that simple systems[4] of three or more coupled non-linear first order equations often admit chaotic solutions called strange attractors. It is this attractor or a few attractors embedded in the state space of the fluid which account for the turbulent behaviour. Motion on these attractors are highly sensitive to

initial conditions and possibly this sensitivity accounts for the stochastic time dependence of the fluid and the unpredictability of the time evolution of the system.

Though turbulence is a very common phenomena in nature we shall be concentrating only on simple physical systems under controlled conditions for the study of onset of turbulence. The dynamical behaviour of some physical systems is dependent strongly on a tunable parameter called the control parameter (r). The system remains in a steady state for low values of r. At $r = r_c$, the critical value, this steady state may be replaced by another steady state or an oscillatory state. Increasing r further, may result in the onset of time dependent motion finally leading the system to a chaotic state. A simple laboratory arrangement for the study of onset of turbulence is the Rayleigh Benard system which is described below.

1.2 Rayleigh Benard system

A fluid of positive expansion coefficient, confined to a rectangular box of depth 'd' is considered. For theoretical simplifications, the lateral dimensions of the box are assumed to extend to infinity. The fluid is heated uniformly from below. The physical conditions of the fluid are described by two dimensionless parameters, the Rayleigh number R

$$R = \frac{\alpha\, g\, d^3 (T_2 - T_1)}{\varkappa\, \nu}$$

and the Prandtl number $\sigma = \frac{\nu}{\varkappa}$

where α = coefficient of volume expansion

g = acceleration due to gravity

$\varkappa$ = coefficient of heat diffusivity

ν = coefficient of kinematic viscosity

T_2 = temperature of lower plate

T_1 = temperature of upper plate

Since the fluid is being heated from below, an adverse temperature gradient exists along the layer which results in a top heavy arrangement. This is a potentially unstable configuration. The natural tendency of the fluid to redistribute itself and attain stability is opposed by its viscosity and conductivity. This stationary state of the fluid however gets destabilized at a critical temperature gradient, characterized by the Rayleigh number R_c. The energy released by the buoyancy force overcomes the dissipation due to the viscous forces and the fluid sets in motion with the hot fluid rising and cold heavy fluid flowing down. Thus for the Rayleigh Benard system at $r = r_c$, we have the onset of the first instability where the stationary conduction state destabilizes against steady state convection. Experimentally it has been observed[9] that the steady state convection state remains stable till about $r = 12\ r_c$. Above this value of r time dependent motion sets in finally leading the system to a chaotic state. To determine this r_c we need to first concentrate on the equations guiding the motion of the fluid.

1.3. The hydrodynamic equation in the Boussinesq approximation

As the convecting fluid moves across the layer, its physical properties like the viscosity and density are likekly to change with the variation in temperature. In the Boussinesq approximation[5], variations of all fluid properties except the density are ignored completely. Variations in density are also ignored except in so far as they give rise to the buoyancy force. The conservation of mass, momentum and heat are expressed respectively as

the continuity equation $\vec{\nabla}\cdot\vec{V} = 0$ (1.1)

the Navier Stokes equation

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v}\cdot\vec{\nabla})\vec{v} = \frac{\vec{\nabla} P}{\rho_o} - \vec{g}\left(1 + \frac{\delta\rho}{\rho_o}\right) + \nu\nabla^2\vec{v} \tag{1.2}$$

where $\delta\rho$ is the variation of density from the equilibrium point. The equation of heat diffusion in the temperature variable T is

$$\frac{\partial T}{\partial t} + (\vec{v}.\vec{\nabla})T = \varkappa\nabla^2 T \tag{1.3}$$

These equations can be reduced to a dimensionless form by scaling

1) all distances by d

2) velocity by $\varkappa/d$

3) time by $d^2/\varkappa$

4) temperature by ΔT ($\Delta T=T_2-T_1$)

We describe the Rayleigh Benard convection in terms of the z component of velocity (w) and θ, the temperature fluctuation from the equilibrium state. If we consider fluids of very high Prandtl number, the non-linear term in the Navier Stokes equation can be

neglected and we arrive at the dimensionless form of the hydrodynamic equations (Appendix A)

$$\nabla^4 w - \nabla^2\left(\frac{\partial w}{\partial t}\right) = -R\,\nabla_1^2\theta \qquad (1.4a)$$

$$\nabla^2\theta - \sigma\frac{\partial \theta}{\partial t} = -w + (\vec{v}\cdot\vec{\nabla})\,\theta \qquad (1.4b)$$

where $\nabla_1^2 = \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}$

The solutions of equations (1.4a) and (1.4b) have to satisfy certain boundary conditions depending on the type of bounding surfaces within which the fluid is confined. Irrespective of the type of boundary we have

$$w = 0 \qquad \text{at } z = 0 \text{ and } d \qquad (1.5a)$$

If the boundaries are rigid we have the no slip condition on the velocity field. This is equivalent to having the x component (u) and the y component (v) of the velocity field to be zero on the boundaries.

$$u = 0\ ,\ v = 0 \qquad (1.5b)$$

Since this condition has to be satisfied for all x and y on the surface we have $\dfrac{\partial w}{\partial z} = 0$ (1.5c)

For stress free boundaries the vanishing of the tangential stress would imply $\dfrac{\partial u}{\partial z} = \dfrac{\partial v}{\partial z} = 0$ (1.5d)

and using the continuity equation we have $\dfrac{\partial^2 w}{\partial z^2} = 0$ (1.5e)

If the bounding surface is a perfect conductor then the temperature fluctuation θ has to be zero on the boundaries.

$$\theta = 0 \tag{1.5f}$$

However, if the boundaries are considered to be thermal insulators, we have

$$\frac{\partial \theta}{\partial z} = 0 \tag{1.5g}$$

The equilibrium state of the system is described by

$$\vec{v} = 0 \tag{1.6a}$$

$$\nabla^2 T = 0 \tag{1.6b}$$

$$T = T_2 - \frac{T_2 - T_1}{d} Z \tag{1.6c}$$

The stability of the equilibrium state (conduction state) has to be tested against infinitesimally small perturbations. For this purpose we employ the technique of linear stability analysis.

1.4 Linear Stability Analysis

In linear stability analysis, the effect of small fluctuations away from a solution to the equations is examined[7] as a function of a parameter (R, the Rayleigh number in our case). The hydrodynamic equations are linearized in the amplitude of the fluctuations and these linearized equations are solved to determine whether the perturbations decay or grow in time. Consequently, the boundaries of a stable region in the parameter space can be defined.

Convective state

Performing linear stability analysis on the conduction state determines the critical Rayleigh number for the destabilization of this stationary state. This takes place via a marginal state which

could be either time independent or oscillatory. For a Rayleigh Benard system with no external constraint, the onset of instability is always via the stationary state[8].

For the laterally infinite system that is considered, the convection pattern can be assumed to be periodic in the x-y plane with a wave number a[9]. The critical wave number a_c is obtained by finding the minimum of R(a). The critical Rayleigh number R_c is obtained by setting $a = a_c$[7] in the expression determined for the critical Rayleigh number.

Linear stability analysis determines the magnitude of the wave number. It provides no information about the horizontal structure of the convective motion[6]. This critical wave number that is determined can be resolved in an infinite number of x and y components and corresponding to each there is a steady state solution. In order to determine the form and stability of the new steady state solutions which evolve from the instability, the non-linear equations have to be considered. They are expanded in a sequence of inhomogeneous equations[11]. From the solvability criteria described in Appendix B, solutions of higher orders are arrived at.

Employing the perturbation technique, Schlüter, Lortz and Busse[12] have concluded that the inclusion of the non-linear terms in the analysis lifts the degeneracy in a_c completely. They have also concluded that the only physically realized form of convection near the threshold is in the form of steady state two dimensional rolls (fig.1). All other forms of the solution of the non-linear equations decay exponentially with time. As mentioned

earlier, the system remains in the steady state over a large range of Rayleigh number and does not exhibit any characteristics of turbulence.

However, if the Rayleigh Benard system is subjected to an external constraint, as for example, rotation about a vertical axis; the stability picture undergoes a complete change. For R slightly greater than R_c, irregular time dependent motion has been observed[13]. This system thus appears to be a simple model exhibiting the characteristics of a weakly turbulent system[3].

In addition to the Rayleigh number and the Prandtl number, the fluid is also described by another dimensionless parameter, the Taylor number $T = 4\, d^4\, \Omega^2 / \nu^2$ where Ω is the angular velocity of the fluid and the other symbols have been defined in the previous sections.

The Navier Stokes equation has to be augmented by the terms for the Coriolis and centrifugal forces while the equation for heat diffusion remains unaltered. The equivalent form of Navier Stokes equation is

$$\frac{\partial \vec{v}}{\partial t} = -\vec{\nabla} \left(\frac{P}{\rho_o} - \frac{1}{2} |\vec{\Omega} \times \vec{r}|^2 \right) + \nu \nabla^2 \vec{v} + 2\, \vec{v} \times \vec{\Omega}$$
$$- \left(1 + \frac{\delta \rho}{\rho_o} \right) \vec{g} \tag{1.7}$$

Using the same scaling factors as for the non-rotating case we arrive at the equations[14] (Appendix A)

$$\nabla^2 \left(\nabla^2 - \frac{\partial}{\partial t} \right) w = \tau \frac{\partial \zeta}{\partial z} - R\, \nabla_1^2\, \theta \tag{1.8a}$$

$$(\nabla^2 - \frac{\partial}{\partial t})\zeta = -\tau \frac{\partial w}{\partial z} \tag{1.8b}$$

$$(\nabla^2 - \sigma \frac{\partial}{\partial t})\theta = -w + (\vec{v}.\vec{\nabla})\theta \tag{1.8c}$$

where $\tau^2 = T$

1.5 Conductive state

The equilibrium state is defined by

$$v = 0 , \quad T = T_s$$

In order to study the stability of this conduction state as in the non-rotating case, we perturb this equilibrium state infinitesimally. The growth of these perturbations are determined by solving the linearized hydrodynamic equations in these fluctuations. An exponential time dependence e^{pt} is assumed for the perturbations. The conduction state would destabilize against these perturbations if Re (p) > 0 . Unlike the non rotating case, the marginal state can be either stationary or oscillatory depending on whether Im (p) = 0 or not for Re (p) = 0. For Im (p) not zero, the instability grows with an oscillating amplitude. However if the analysis is confined to fluids of large Prandtl numbers, the onset of instability is always stationary. Oscillatory instability is always preferred for fluids with Prandtl number less than 0.67 [7]. For fluids with high Prandtl number since the oscillatory state is ruled out in the non-linear region, we have a direct transition to a chaotic state[18].

Linear stability analysis predicts that the conduction state is destabilized in favour of steady state convection at a critical Rayleigh number which is dependent on the Taylor number.

Rotation thus has an inhibiting effect on the onset of convection[7] implying that any finite value of T increases R beyond that calculated for the non rotating case.

As in the non rotating case, linear stability analysis predicts just the magnitude of the critical wave number. Due to the horizontal isotropy of the system all directions are equally preferred. Thus an infinite number of solutions all with the same wave number but differing in their orientation is possible. For determining the physically realized solution, inclusion of the non-linear terms in the equations is essential. Employing the perturbation technique, Schlüter, Lortz and Busse[11] and Küppers and Lortz[2] have determined that two dimensional rolls represent the only form of stable solutions for the rotating case as well. However, unlike the non rotating case, these steady state two dimensional rolls become unstable if the Taylor number exceeds a critical value T_c[10]. This happens for R just above the threshold. An investigation of this instability of the convection rolls for slightly super-critical Rayleigh number is the key to studying the Küppers Lortz instability which is discussed in the next section.

1.6 Küppers Lortz Instability

Rayleigh Benard convection in a rotating horizontal layer develops an interesting complication just above the threshold. Küppers and Lortz[2] have discovered that a set of cylindrical rolls formed at the onset of convection destabilizes in favour of another set developing at an angle to the original roll provided the Taylor number is above the critical value T_c. These

perturbing rolls do not grow indefinitely. They in turn sucuumb to a similar type of instability. Thus we have a situation in which the solutions switch over from one roll system to another[3]. This transition from one set to another is nearly periodic with a transition time that fluctuates statistically about a mean value thus giving rise to weak turbulence.

SECTION II

OVERVIEW

In the chapters that follow we investigate the Küppers Lortz instability in horizontal fluids using truncated models. While the linearized hydrodynamic equations are adequate for studying the onset of convection, the non-linear equations are essential for tracing the course of the system after the onset. The coupled, non-linear partial differential equations are difficult to handle. With the discovery by Lorenz[8] that the essential information is contained in just a few modes of the system, it has been possible to consider the non-linear effects in a simplified manner. Using a truncated representation one can reduce[14] the partial differential equations to a system of coupled non-linear ordinary differential equations. The truncated system provides an adequate description of the onset of convection and can be numerically integrated easily (Appendix C) facilitating the study of non linear effects near the threshold.

Chapter II deals with the Küppers Lortz instability in fluids with infinite Prandtl number. We have first considered a minimal representation of the velocity, vorticity and temperature modes. This has the advantage of being handled analytically and hence the qualitative features of Küppers Lortz instability can be explored in a simplified manner using such a model. However for more accurate results, we need to consider a more complete model by

considering higher order terms in the expansion of the velocity, vorticity and temperature fields.

In chapter III we consider Küppers Lortz instability in electrically conducting fluids in the presence of a magnetic field. Acting independently, both magnetic field and rotation have an inhibiting effect on the onset of convection[7]. However, when the fluid is subjected to these two constraints simultaneously, the effect is to reduce the threshold Rayleigh number below the simple superposition of the two effects acting individually. Thus an interesting question is how would the magnetic field affect the Küppers Lortz instability. We find that it always stabilizes the system against this instability. In fact the critical Taylor number is found to be a monotonically increasing function of the magnetic field. Presence of magnetic field also enables us to investigate Küppers Lortz instability in low Prandtl number fluids. In the absence of magnetic field for Taylor number sufficiently high, the onset is oscillatory if the Prandtl number is less than 0.67. However if it is subjected to just the magnetic field the onset as stationary convection is ensured under all terrestrial conditions. If the magnetic field and rotation are present simultaneously there exists a range of magnetic field for which the onset is stationary[7] thus making it possible to investigate Küppers Lortz instability in low Prandtl number fluids particularly in liquid metals like Mercury (σ = 0.025)

Finally in chapter IV we discuss the case of a rotating fluid layer between two thermally insulating boundaries. We determine

the critical Rayleigh number, the critical wave number and address the question of Küppers Lortz instability. We find that for a range of rotation speeds, the onset of stationary convection is in the form of long wavelength rolls. An investigation of Küppers Lortz instability shows that a set of rolls grow at right angles to the original set at very low Taylor numbers indicating presumably the preference of square rolls over the cylindrical rolls.

As we increase the rotation speed, the rolls become unstable for a range of angles and finally for very large T the basic rolls would be unstable to rolls at all angles.

CHAPTER - II

KÜPPERS LORTZ INSTABILITY IN FLUIDS WITH LARGE PRANDTL NUMBER

2.1 The Lorenz model for Rayleigh Benard convection

A layer of fluid of depth d and infinite lateral dimensions is heated from below such that an adverse temperature gradient is maintained across the layer. Though this represents an unstable configuration, the viscosity and conductivity of the fluid tend to stabilize the system and the fluid remains stationary. As the temperature gradient across the layer increases, the buoyancy force overcomes the dissipative effects of viscosity and motion sets in. This critical temperature gradient (defined by the Rayleigh number) also depends on the physical properties of the fluid.

$$R = \frac{\alpha\, g\, d^3\, \Delta T}{\varkappa\, \nu}$$

where α is the coefficient of volume expansion

g is the acceleration due to gravity

ΔT is the temperature difference across the layer

$\varkappa$ is the coefficient of thermal diffusivity

ν is the coefficient of kinematic viscosity

The motion of the convective fluid is guided by the Navier Stokes equation which in the Boussinesq approximation is

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \vec{\nabla}) \vec{v} = - \frac{\vec{\nabla} P}{\rho_o} + \nu \nabla^2 \vec{v}$$

$$- (1 + \frac{\delta \rho}{\rho_o}) \vec{g} \qquad (2.1)$$

where P represents the pressure, $\delta\rho$ is the variation of density from the steady conductive state. Considering T as the temperature variable, the equation for heat diffusion is

$$\frac{\partial T}{\partial t} + (\vec{v} \cdot \vec{\nabla}) T = \varkappa \nabla^2 T \qquad (2.2)$$

The equation of state is

$$\rho = \rho_o \{ 1 + \alpha (T_2 - T) \} \qquad (2.3)$$

Where ρ_o is the density of the fluid at the lower plate which is at temperature T_2.

We assume the fluid to be surrounded by free surfaces and perfectly conducting boundaries. Though the assumption of the boundaries being free surfaces is unphysical, nevertheless it helps in determining the qualitative features of the flow in a simplified manner. Rigid boundaries are more likely to be encountered in the experiments and hence are more realistic but they have the disadvantage of involving a lot of mathematical complications. In most of our analysis we shall confine ourselves to free boundaries . In Appendix D we perform our calculations for the rigid boundaries.

The conduction state of the fluid is described by a linear temperature profile and $\vec{v} = 0$. When the fluid starts convecting

this linear temperature profile gets distorted due to the motion of the fluid . If θ is assumed to be this deviation, the altered temperature is given as

$$T = T_2 - \frac{T_2 - T_1}{d} Z + \theta \tag{2.4}$$

With δP as the change in pressure, the equation of the convecting fluid becomes

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v} . \vec{\nabla}) \vec{v} = - \vec{\nabla} (\frac{\delta P}{\rho_o}) + \vec{g} \alpha \theta + \nu \nabla^2 \vec{v} \tag{2.5a}$$

$$\frac{\partial \theta}{\partial t} - \varkappa \nabla^2 \theta = (\vec{v} . \vec{\nabla}) \theta - w \tag{2.5b}$$

Performing the same scalings as in Chapter I, we arrive at the dimensionless form of equations (2.5a) and (2.5b). In Appendix-A we reduce these equations to a scalar form. Thus the system is described by w, the z component of velocity and θ, the distortion in the temperature field. The relevant equations are

$$\nabla^2 (\nabla^2 - \frac{\partial}{\partial t}) w = -R \nabla_1^2 \theta \tag{2.6a}$$

$$(\nabla^2 - \sigma \frac{\partial}{\partial t}) \theta = - w + (\vec{v} . \vec{\nabla}) \theta \tag{2.6b}$$

The solutions of these equations in general are represented in terms of the complete set of normal modes of the system. Lorenz truncation[8] refers to expressing these solutions in terms of only a few modes of the system. For the present problem these modes

are the Fourier modes. The simplest expansions for w and θ, compatible with the boundary conditions (stress-free) are

$$w = a(t) \cos a x \sin \pi z \tag{2.7a}$$

$$\theta = b(t) \cos a x \sin \pi z + c(t) \sin 2\pi z \tag{2.7b}$$

Substitution of the expansions (2.7a) and (2.7b) in equations (2.6a) and (2.6b) results in a set of ordinary differential equations which are called the Lorenz equations. With X , Y and Z representing the scaled versions of a(t), b(t) and c(t) respectively the dynamical system that is obtained is[8]

$$\dot{X} = \sigma (- X + Y) \tag{2.8a}$$

$$\dot{Y} = - X Z + r X - Y \tag{2.8b}$$

$$\dot{Z} = X Y - b Z \tag{2.8c}$$

where $b = \dfrac{4 \pi^2}{(\pi^2 + a^2)}$ and $r = \dfrac{R}{R_c}$

We rescale time by a factor of σ.

In these equations, X is proportional to the intensity of convective flow, Y represents the temperature difference between the ascending and descending currents. The variable Z is proportional to the distortion of the temperature profile from the linear distribution.

2.2 Linear Stability Analysis

The steady state solution of equations (2.8a)-(2.8c) is obtained by considering all time derivatives to be zero. In

general we consider the steady state solution to be represented by $(X_o Y_o Z_o)$. To determine the stability of this steady state we superimpose infinitesimal perturbations $(x\ y\ z)$ on these solutions and study the time evolution of these perturbations by setting the hydrodynamic equations in them, neglecting all non-linear terms. If a time dependence of the form e^{pt} is assumed, the steady state would be destabilized by these disturbances if Re (p) is greater than zero. These perturbations are governed by the linearized equations which are

$$\dot{x} = \sigma (- x + y) \tag{2.9a}$$

$$\dot{y} = (r - Z_o) x - y - X_o z \tag{2.9b}$$

$$\dot{z} = Y_o x + X_o y - b z \tag{2.9c}$$

For a non trivial solution of these equations, the determinant of the matrix formed by the coefficient of x,y,z must necessarily be zero. Recalling that the perturbations have a time dependence of the form e^{pt} the solvability condition is given by the characteristic equation of the matrix [8].

$$(p + b)\{ p^2 + p (1 + \sigma) + \sigma (1 - r) \} = 0 \tag{2.10}$$

with $(X_o Y_o Z_o) = 0$ for the conduction state.

For $r > 0$, the equation has three real roots. If $r < 1$, all the roots are less than zero and the perturbations decay with time. For $r > 1$, one of these roots becomes positive and thus the conduction state gets destabilized at $r = 1$ ($R = R_c$). For $r > 1$ we have two additional steady state solutions given by

$$X = Y = \pm \{ b (r - 1) \}^{0.5}$$

$$Z = (r - 1)$$

For both these solutions, the characteristic equation of the matrix is

$$p^3 + p^2 (b + 1 + \sigma) + p b (r + \sigma) + 2 b \sigma (r - 1) = 0$$

For r > 1 , this equation possesses one real negative root and two complex conjugate roots.

If the real root is p_o and the complex conjugate roots are represented by $\pm$ i ω_o we have

$$(p - p_o) (p - i \omega_o) (p + i \omega_o) = 0 \qquad (2.12a)$$

$$\text{or} \quad p^3 - p^2 p_o + \omega_o^2 p - \omega_o^2 p = 0 \qquad (2.12b)$$

Thus for the roots to be pure imaginary the constant term is the product of the coefficients of p^2 and p. Applying this condition to equation (2.11) we obtain the critical r for the onset of oscillatory instability as

$$r_{os} = \frac{\sigma (b + 3 + \sigma)}{(\sigma - b - 1)} \qquad (2.13)$$

If $\sigma < (b + 1)$, the steady state convection rolls will always remain stable. For $\sigma > (b + 1)$ at $r = r_{os}$ these rolls are destabilized and oscillatory instability sets in[8].

2.3 Rayleigh Benard convection in a rotating fluid

If the Rayleigh Benard system is rotated about a vertical axis , the region of stability gets considerably altered. Unlike the non rotating case, the marginal state can now be oscillatory provided the Prandtl number is less than 0.67. Another

characteristic feature of the rotating system is the appearance of time dependent motion immediately above the threshold. Küppers and Lortz[2] have determined that among all the possible steady state solutions just above the threshold,only convection in the form of two dimensional rolls are stable. A similar observation was made by Schlüter, Busse and Lortz[11] for the non rotating case. However for the rotating case, above a critical rotation speed characterized by the dimensionless parameter T, the Taylor number, these steady state two dimensional rolls are destabilized and no stable stationary solution is possible. In the limit of large Prandtl number, since oscillatory instabilities are precluded, the physically realized convection presents an enigmatic problem. According to Küppers and Lortz, the primary cause of the restricted stability domain of the two dimensional convection arises from a non linear effect. The cylindrical rolls that are formed at the onset of convection get destabilized by another set of rolls which develop at an angle to the original set if the rotation speed is above the critical value. This instability which occurs right at the threshold is known as the Kuppers Lortz instability . For fluids with large Prandtl number the critical Taylor number determined was 2285 with the assumption of free boundaries.

2.4 Küppers Lortz instability

An investigation of the time evolution of the Küppers Lortz instability gives an insight into the origin of the irregular

time dependent motion and the transition to a chaotic state above the threshold .We represent the steady state rolls by the expansions of the form[18].

$$w = f(z, a_c) \sum_{n=-N}^{N} C_n(t)\, e^{i \vec{k}_n \cdot \vec{r}} \qquad (2.14)$$

for the velocity field and similar expansions for the vorticity and temperature modes .Here a_c refers to the critical wave number.The c_n' s are assumed to be weakly time dependent . The horizontal vectors $\vec{k}_n$' s are arbitrary except that $|\vec{k}_n| = a_c$,

$k_{-n} = - k_n$.

We start with the initial value for C_1 given by the steady state solution for the two dimensional rolls and much smaller values for all other coefficients C_n (fig. 2)[3]. The coefficient C_2 corresponding to the vector $\vec{k}_2$ grows until it reaches an amplitude comparable to that of C_2 at which point , C_2 starts to decay and is eventually replaced by C_2 as the approximately steady state solution . At the same time , the disturbance amplitude C_3 corresponding to $\vec{k}_3$ starts growing exponentially . After a while the amplitude of C_3 becomes comparable to that of C_2 and finally C_2 is replaced by C_3. Since C_3 had been decaying before it started growing,it takes longer for this disturbance to reach the equilibrium value. This process is repeated cyclically but the period tends to grow without limit. This however is an unphysical feature of the solution.The properties of the system appear to depend on the time elapsed since the initial conditions were set.

This inhomogeneity in time, of the solution, can be resolved by realizing that the disturbances are present at all times and not just at the initial moment as assumed . The presence of noise level[11] prevents the amplitudes from decaying to arbitrarily low levels. At the same time it introduces a random element into the time dependence of the system. Thus the weakly non-linear problem of convection in rotating system exhibits a typical feature of turbulent fluids in that its time evolution is stochastic in nature.

2.5 Hydrodynamic equations for the rotating system

Rayleigh Benard convection for rotating fluids is described in terms of three dimensionless variables w , the z component of velocity, ζ ,the z component of vorticity and θ,the deviation of the temperature from the conduction state. The motion of the convecting fluid is guided by the usual Navier Stokes equation augmented by the Coriolis and centrifugal forces . Thus we have

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \vec{\nabla}) \vec{v} = - \vec{\nabla} \left(\frac{P}{\rho_o} - \frac{1}{2} |\vec{\Omega} \times \vec{r}|^2 \right)$$

$$+ \nu \nabla^2 \vec{v} + 2 \vec{v} \times \vec{\Omega} - \vec{g} \left(1 + \frac{\delta \rho}{\rho_o} \right) \qquad (2.15a)$$

The equation for heat diffusion remains unaltered by the inclusion of the effects of rotation . Thus we have

$$\frac{\partial T}{\partial t} + (\vec{v} \cdot \vec{\nabla}) T = \kappa \nabla^2 T \qquad (2.15b)$$

As derived in appendix A , considering the curl on both sides of equation (2.15a) and taking its z component results in the equation for ζ .We therefore have

$$\frac{\partial \zeta}{\partial t} = 2 \Omega \frac{\partial w}{\partial z} + \nu \nabla^2 \zeta \qquad (2.15c)$$

Using the results of Appendix A and the scaling factors of chapter I,with ζ scaled by κ / d^2, we arrive at the following dimensionless form of the equations.

$$\nabla^2 (\nabla^2 - \frac{\partial}{\partial t}) w = \tau \frac{\partial \zeta}{\partial z} - R \nabla_1^2 \theta \qquad (2.16a)$$

$$(\nabla^2 - \frac{\partial}{\partial t}) \zeta = - \tau \frac{\partial w}{\partial z} \qquad (2.16b)$$

$$(\nabla^2 - \sigma \frac{\partial}{\partial t}) \theta = - w + (\vec{v} \cdot \vec{\nabla}) \theta \qquad (2.16c)$$

We represent the steady state cylindrical rolls along the y axis in the form of

$w = a (t) \cos ax \sin \pi z$

$\zeta = f (t) \cos ax \cos \pi z$

$\theta = b (t) \cos ax \sin \pi z + c (t) \sin 2\pi z$

Substitution of these expansions in equations (2.16a)-(2.16c) lead to a set of Lorenz equations similar to the non rotating case. Thus the dynamical system that we obtain is

$$\dot{X} = \sigma (- X + Y + G t) \qquad (2.17a)$$

$$\dot{Y} = - X Z + r X - Y \qquad (2.17b)$$

$$\dot{Z} = X Y - b Z \qquad (2.17c)$$

$$\dot{G} = - \sigma (G + t X) \qquad (2.17d)$$

where $b = \dfrac{4 \pi^2}{\pi^2 + a^2}$, $t^2 = \dfrac{\pi^2 T}{(\pi^2 + a^2)^3}$

We scale time by σ .

The steady state solution of this system of equations is given by $X = Y = Z = G = 0$ .Here X, Y, Z and G are the scaled forms of a, b, c and f respectively. We perform linear stability analysis on this state to determine the critical Rayleigh number and the critical wave number for the onset of convection . Assuming perturbations x,y,z and g with time dependence of the form e^{pt} and realizing that the equation for z decouples when we linearize in these perturbations , we obtain the condition

$$r = 1 + t^2$$

for the onset of stationary convection . The critical wave number a_c is obtained by minimizing R(a) with respect to a. This leads to the condition $2 m^3 + 3 m^2 = 1 + \dfrac{T}{\pi^4}$ (2.18)

where $m = \left(\dfrac{a_c}{\pi}\right)^2$

Substitution of $a = a_c$ in the expression for R determines the critical Rayleigh number.

$$R_c = \frac{(\pi^2 + a_c^2)^3 + T\pi^2}{a_c^2} \tag{2.19}$$

2.6 Küppers Lortz instability with truncated model

The cylindrical rolls that are formed at the onset are represented as

$$Y_o = (1 + t^2) X_o \tag{2.20a}$$

$$G_o = -t X_o \tag{2.20b}$$

$$Z_o = r - (1 + t^2) \tag{2.20c}$$

Investigation of the stability of these convection rolls forms the basis for the study of Küppers Lortz instability. These rolls are stable against self perturbations,that is,by the rolls whose axis is along the y direction . We superimpose another set of rolls , the axis of which is in an arbitrary direction and study the time development of these rolls. The velocity, vorticity and temperature modes would then have y dependence as well. Accordingly we represent them by

$$w = \sin \pi z \{ a(t) \cos ax + a_1(t) \cos (k_1 x + k_2 y) \} \tag{2.21a}$$

$$\zeta = \cos \pi z \{ f(t) \cos ax + f_1(t) \cos (k_1 x + k_2 y) \} \tag{2.21b}$$

$$\begin{aligned} \theta = \sin \pi z \{ b(t) \cos ax + b_1(t) \cos (k_1 x + k_2 y) \} \\ + \sin 2\pi z \{ c(t) + c_1(t) \cos (k_1 x + a x + k_2 y) + \\ c_2(t) \cos (k_1 x + k_2 y - a x) \} \end{aligned} \tag{2.21c}$$

The above expansions are consistent with the boundary conditions for stress free, perfectly conducting surfaces with

$$w = \frac{\partial^2 w}{\partial z^2} = 0 \ , \quad \frac{\partial \zeta}{\partial z} = 0 \ , \quad \theta = 0 \qquad (2.22)$$

Introducing these expansions in equations (2.21a)-(2.21c) we obtain the corresponding Lorenz model. Scaling time by σ and

$$b_{\pm} = \frac{4\pi^2 + 2a^2(1 \pm \cos\theta)}{(\pi^2 + a^2)} \quad ; \qquad c = \frac{(\pi^2 + a^2)^{0.5}}{\pi}$$

we have

$$\dot{X} = \sigma(-X + Y + tG) \qquad (2.23a)$$

$$\dot{Y} = -XZ + rX - Y - \frac{Z_1}{4}\left\{(1 - \cos\theta)X_1 + G_1 c \sin\theta\right\} - \frac{Z_2}{4}\left\{(1 + \cos\theta)X_1 - G_1 c \sin\theta\right\} \qquad (2.23b)$$

$$\dot{Z} = XY + X_1Y_1 - bZ \qquad (2.23c)$$

$$\dot{G} = -\sigma(G + tX) \qquad (2.23d)$$

$$\dot{X}_1 = \sigma(-X_1 + Y_1 + tG_1) \qquad (2.23e)$$

$$\dot{Y}_1 = -X_1 Z + rX_1 - Y_1 - \frac{Z_1}{4}\{(1 - \cos\theta)X$$

$$- G\, c \sin \theta \,\} - \frac{Z_2}{4} \{ (1 + \cos \theta) X + G\, c \sin \theta \} \tag{2.23f}$$

$$\dot{Z}_1 = \frac{(1 - \cos \theta)}{2} (X Y_1 + X_1 Y) - b_+ Z_1 + \frac{c \sin \theta}{2} (G_1 Y - G Y_1) \tag{2.23g}$$

$$\dot{Z}_2 = \frac{(1 + \cos \theta)}{2} (X Y_1 + X_1 Y) - b_- Z_2 + \frac{c \sin \theta}{2} (G Y_1 - G_1 Y) \tag{2.23h}$$

$$\dot{G}_1 = - \sigma (X_1 + G_1) \tag{2.23i}$$

The two sets of rolls are given either by X,Y, Z and G non zero and the others zero or by X_1, Y_1, Z and G_1 non zero and the others zero. We assume the basic pattern to be the rolls along the y axis which is given by the former conditions and the other set is assumed to be representing the perturbing set. Linearizing in these perturbations, we set up the hydrodynamic equations with the assumption that they have an exponential time dependence. The condition for the onset of Küppers Lortz instability is given by $p = 0$.

The rolls along an arbitrary direction are described by

$$Y_1 = (1 + t^2) X_1 , \quad G_1 = - t X_1 \tag{2.24a}$$

After dropping all time derivatives, we obtain the expressions for Z_1 and Z_2 as

$$b_+ Z_1 = (1 - \cos\theta)(1 + t^2)\ X X_1 \tag{2.24b}$$

$$b_- Z_2 = (1 + \cos\theta)(1 + t^2)\ X X_1 \tag{2.24c}$$

The condition for Küppers Lortz instability is given by considering $\dot{Y}_1 = 0$. This is equivalent to

$$\frac{Z_1}{4}(1 - \cos\theta + c\,t\sin\theta) + \frac{Z_2}{4}(1 + \cos\theta - c\,t\sin\theta) = 0 \tag{2.25}$$

where we have used (2.20a)-(2.20c) and (2.24a). Substitution of (2.24b) and (2.24c) results in the determination of the domain of Küppers Lortz instability . The critical Taylor number is obtained as[19]

$$c\,t\sin\theta = \frac{(1 + \cos\theta)^2\, b_+ + (1 - \cos\theta)^2\, b_-}{(1 + \cos\theta)\, b_+ - (1 - \cos\theta)\, b_-} \tag{2.26}$$

Substitution of c ,t , b_+ and b_- results in the critical wave number as a_c as $(\sqrt{3})\pi$ and corresponding to this we have T_c as[19] $80\pi^4$. Though this result is somewhat higher than that calculated by Küppers and Lortz but the minimal model is useful in determining the qualitative features of Küppers Lortz instability . The critical angle of the perturbing rolls predicted by this model is 61°. For more accurate results , we consider a higher order truncated model for the velocity and vorticity modes. Accordingly in w we have the additional terms of the form

$$\sin 2\pi z \, \{a_2(t)\cos(k_1 x + ax + k_2 y) + a_3(t)\cos(k_1 x - ax + k_2 y)\}$$

Similarly the additional terms in the expression for ζ are

$$\cos 2\pi z\{f_2(t)\cos(k_1 x + ax + k_2 y) + f_3(t)\cos(k_1 x - ax + k_2 y)\}$$

The expansion for θ remains unaltered and is given by equation (2.21c) . Inclusion of higher order terms in the expansion of w and ζ gives rise to a more complicated Lorenz model. Our dynamical system now contains a set of 13 ordinary differential equations. The additional equations are

$$\dot{X}_2 = \sigma \left\{ \frac{2 \ t \ G_2}{b_+} - \frac{2 \ Z_1(1 + \cos \theta)}{b_+} - b_+ \ X_2 \right\} \tag{2.27a}$$

$$\dot{X}_3 = \sigma \left\{ \frac{2 \ t \ G_3}{b_-} - \frac{2 \ Z_2 \ (1 - \cos \theta)}{b_-} - b_- \ X_3 \right\} \tag{2.27b}$$

$$\dot{G}_2 = - \sigma \ (2 \ t \ X_2 + b_+ \ G_2) \tag{2.27c}$$

$$\dot{G}_3 = - \sigma \ (2 \ t \ X_3 + b_- \ G_3) \tag{2.27d}$$

The equations for $\dot{Y}$, $\dot{Y}_1$, $\dot{Z}_1$, $\dot{Z}_2$ are altered due to the additional terms in the velocity and the vorticity modes. However,the equations for $\dot{X}$, $\dot{G}$, $\dot{X}_1$, $\dot{G}_1$ and $\dot{Z}$ remain unaltered and are given respectively by (2.23a), (2.23d), (2.23e), (2.23i) and (2.23c). The altered equations are

$$\dot{Y} = -XZ + rX - Y - \frac{Z_1}{4}\left\{(1-\cos\theta)X_1 + c\sin\theta\, G_1\right\} - \frac{Z_2}{4}\left\{(1+\cos\theta)X_1 - c\sin\theta\, G_1\right\}$$

$$- \frac{Y_1}{8}\, c\sin\theta \left\{\frac{G_3}{(1-\cos\theta)} - \frac{G_2}{(1+\cos\theta)}\right\} \tag{2.27e}$$

$$\dot{Y}_1 = -X_1 Z + rX_1 - Y_1 - \frac{Z_1}{4}\left\{(1-\cos\theta)X - c\sin\theta\, G\right\} - \frac{Z_2}{4}\left\{(1+\cos\theta)X + c\sin\theta\, G\right\}$$

$$+ \frac{Y}{8}\, c\sin\theta \left\{\frac{G_3}{(1-\cos\theta)} - \frac{G_2}{(1+\cos\theta)}\right\} \tag{2.27f}$$

$$\dot{Z}_1 = \frac{(1-\cos\theta)}{2}(X_1 Y + X Y_1) - b_+ Z_1 - r X_2$$

$$+ \frac{c\sin\theta}{2}(G_1 Y - G Y_1) \tag{2.27g}$$

$$\dot{Z}_2 = \frac{(1+\cos\theta)}{2}(X_1 Y + X Y_1) - b_- Z_2 - r X_3$$

$$+ \frac{c\sin\theta}{2}(G Y_1 - G_1 Y) \tag{2.27h}$$

The fixed point representing rolls along the y axis are now represented by X,Y,Z and G as non zero and the other coefficients as zero . In this case study of Küppers Lortz instability is equivalent to studying the growth of the perturbations

(X_1, Y_1, G_1, Z_1, Z_2, X_2, G_2,X_3, G_3). As in the earlier cases we again assume these perturbations to be exponentially dependent on time. For the onset of instability in the form of stationary convection ,we determine the relationship between T and θ which makes p = 0 .The rolls along the arbitrary direction are represented by

$$Y_1 = (1 + t^2) X_1 \quad , \quad G_1 = - t X_1 \quad , \quad G_2 = - \frac{2 t X_2}{b_+}$$

$$G_3 = - \frac{2 t X_3}{b_-} \tag{2.28}$$

Using equations (2.20a)- (2.20c) ,(2.27a),(2.27b),(2.27g) and (2.27h) and considering all time derivatives to be zero in these terms we arrive at the expression for $Z_1$and Z_2 as

$$b_+ Z_1 = \frac{(1 - \cos \theta) X X_1 (1 + t^2) (b_+^3 + 4 t^2 b_+)}{b_+^3 + 4 t^2 b_+ - 2 r (1 + \cos \theta)} \tag{2.29a}$$

$$b_- Z_2 = \frac{(1 + \cos \theta) X X_1 (1 + t^2) (b_-^3 + 4 t^2 b_-)}{b_-^3 + 4 t^2 b_- - 2 r (1 - \cos \theta)} \tag{2.29b}$$

The condition for the onset of Küppers Lortz instability is given by considering $\dot{Y}_1 = 0$.

$$p = (1 - \cos\theta + c\,t\sin\theta)\,Z_1 + (1 + \cos\theta - c\,t\sin\theta)\,Z_2$$

$$+ (1 + t^2)\,c\,t\sin\theta \left\{ \frac{X_2}{2\,b_+(1+\cos\theta)} - \frac{X_3}{2\,b_-(1-\cos\theta)} \right\} \tag{2.30}$$

Considering $\theta = 0$ in equation (2.30) results in a negative value of p which signifies that the perturbations coincident with the basic pattern decay with time and hence the original set remains stable to self- perturbations. The onset of Küppers Lortz instability is given by putting $p = 0$. A plot of T against θ determines the domain of Küppers Lortz instability in the T-θ space. The minimum of such a plot corresponds to the critical Taylor number at which rolls along the y axis are destabilized by rolls at an angle of about 58°. The minimum so determined is approximately $23\,\pi^4$. For T greater than this value, the original rolls become unstable to perturbations within a certain range of angles with the range increasing with the rotation speed (fig.3).

CHAPTER-III

KÜPPERS LORTZ INSTABILITY IN PRESENCE OF A MAGNETIC FIELD

3.1 Introduction

Rayleigh - Benard convection in a conducting fluid in the presence of magnetic field and rotation has an interesting feature. Acting independently, both rotation and magnetic field have an inhibiting effect on the onset of convection. When present simultaneously, the effect is to lower the threshold below the simple superposition. In this chapter we investigate the Küppers - Lortz instability in fluids of high Prandtl number in the presence of magnetic field.

3.2 Hydrodynamic Equations

The Navier Stokes equation has to be augmented by the Lorentz force arising due to the magnetic field. The effects of displacement current are consistently ignored.

We have

$$\dot{\vec{v}} + (\vec{v}\cdot\vec{\nabla}) \vec{v} = - \vec{\nabla} \left[\frac{P}{\rho_o} - \frac{1}{2} | \vec{\Omega} \times \vec{r} |^2 + \frac{\mu}{8\pi\rho_o} H^2 \right] - (1 + \frac{\delta \rho}{\rho_o}) \vec{g}$$

$$+ \nu \nabla^2 \vec{v} + 2 \vec{v} \times \vec{\Omega} + \frac{\mu}{4\pi\rho_o} (\vec{H}\cdot\vec{\nabla}) \vec{H} \qquad (3.1)$$

$\delta\rho$ is the variation of density from the conduction state value, Ω is the rotation speed along the z - direction. $\vec{H}$ is the magnetic field in the fluid, μ is the permeability and ν is the kinematic viscosity. As in the earlier cases, Boussinesq approximation is assumed.

As the heat diffuses, the temperature T changes at various points. The diffusion equation for the temperature variable is

$$\frac{\partial T}{\partial t} + (\vec{v} \cdot \vec{\nabla})T = \varkappa \nabla^2 T \tag{3.2}$$

with $\varkappa$ as the thermal diffusivity.

The equation of motion for the magnetic field is derived using the Maxwell's equations. Also a fluid with velocity $\vec{v}$, placed in a magnetic field $\vec{H}$, experiences an electric field[15]

$$\vec{E} + \mu \vec{v} \times \vec{H}$$

The current density

$$\vec{J} = \sigma_1 (\vec{E} + \mu \vec{v} \times \vec{H}) \tag{3.3}$$

Hence, $\vec{E} = \frac{\vec{J}}{\sigma_1} - \mu \vec{v} \times \vec{H}$

Using $$\text{Curl } \vec{H} = 4 \pi \vec{J} \tag{3.4}$$

we have $$\vec{E} = \frac{\text{Curl } \vec{H}}{4 \pi \sigma_1} - \mu \vec{v} \times \vec{H} \tag{3.5}$$

Taking Curl on both sides of equation (3.5) we obtain

$$\text{Curl } \vec{E} = \text{Curl} \left[\frac{\text{Curl } \vec{H}}{4 \pi \sigma_1} \right] - \mu \text{ Curl } (\vec{v} \times \vec{H}) \tag{3.6}$$

From Maxwell's equations we have

$$\text{Curl } \vec{E} = - \mu \frac{\partial \vec{H}}{\partial t} \tag{3.7a}$$

$$\text{div } \vec{H} = 0 \tag{3.7b}$$

Using (3.7) in (3.6) we obtain

$$- \mu \frac{\partial \vec{H}}{\partial t} = - \frac{\nabla^2 \vec{H}}{4 \pi \sigma_1} - \mu \text{ Curl } (\vec{v} \times \vec{H}) \tag{3.8}$$

$$\frac{\partial \vec{H}}{\partial t} + (\vec{v} \cdot \vec{\nabla}) \vec{H} = \eta \nabla^2 \vec{H} + (\vec{H} . \vec{\nabla}) \vec{v} \tag{3.9}$$

where $\eta = \dfrac{1}{4 \pi \mu \sigma_1}$

The steady state is defined by $\vec{v} = 0$

$$T = T_s = T_2 - \frac{T_2 - T_1}{d} Z$$

$$\vec{H} = \vec{H}_o$$

with $\vec{H}_o$ as the constant magnetic field that the fluid is subjected to. For mathematical simplifications, we assume $\vec{H}_o$ to be along the z direction.

As in chapter II we scale

i) all distances by `d ʹ

hi) time by d^2/ ν

iii) temperature by ΔT

iv) velocity by $d/ \varkappa$

v) magnetic field by $\frac{\varkappa}{\eta} H_o$ [16]

with $T = \tau^2 = \dfrac{4 \Omega^2 d^4}{\nu^2}$, $\nabla_1^2 = \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}$

and $Q = \dfrac{\mu H_0^2 d^2}{4 \pi \rho \mu \nu}$

the equations in the dimensionless variables reduce to

$$\nabla^2 \left(\frac{\partial}{\partial t} - \nabla^2 \right) w = R \nabla_1^2 \theta - \tau \frac{\partial \zeta}{\partial z} + Q \nabla^2 \frac{\partial h}{\partial z} + \frac{1}{\sigma} \left[\vec{\nabla} \times \vec{\nabla} \times (\vec{v} \cdot \vec{\nabla}) \vec{v} \right]_z \tag{3.10}$$

$$\left[\frac{\partial}{\partial t} - \nabla^2 \right] \zeta = \tau \frac{\partial w}{\partial z} + Q \frac{\partial J}{\partial z} + \frac{1}{\sigma} \left[\vec{\nabla} \times (\vec{v} \cdot \vec{\nabla}) \vec{v} \right]_z \tag{3.11}$$

$$\left[\sigma \frac{\partial}{\partial t} - \nabla^2 \right] \theta = w - (\vec{v} \cdot \vec{\nabla}) \theta \tag{3.12}$$

$$\left[\sigma_z \frac{\partial}{\partial t} - \nabla^2 \right] h = \frac{\partial w}{\partial z} - \frac{\varkappa}{\eta} \left\{ (\vec{v} . \vec{\nabla}) h - (\vec{H} . \vec{\nabla}) w \right\} \tag{3.13a}$$

$$\left[\sigma_z \frac{\partial}{\partial t} - \nabla^2 \right] j = \frac{\partial \zeta}{\partial z} - \frac{\varkappa}{\eta} \left\{ (\vec{v} . \vec{\nabla}) j - (\vec{H} . \vec{\nabla}) \zeta \right\} \tag{3.13b}$$

The strength of the non-linear terms are now discussed. In equations (3.1) and (3.13) the non-linear ($\vec{H} \cdot \vec{\nabla}$) $\vec{H}$ term is of O ($\frac{\sigma_z}{\sigma}$) as also the non-linearities ($\vec{v} . \vec{\nabla}$) $\vec{H}$ and ($\vec{H} . \vec{\nabla}$) $\vec{v}$. Under all terrestrial conditions $\sigma_z / \sigma \ll 1$ and therefore the above non-linear terms can be safely ignored. However the non-linear term in the equation for θ is O (1) therefore it has to be retained. In this chapter we consider fluids with infinite Prandtl number, thus the ($\vec{v} . \vec{\nabla}$) $\vec{v}$ term can also be ignored since it is of the O (σ^{-1}).

Defining $\vec{J} = \vec{\nabla} \times \vec{H}$ and for the z-components of velocity, vorticity, magnetic field and current density we use the notations w, ζ, h and j respectively.

Assuming the time dependence of the form e^{pt}, for the perturbations $\vec{v}$, θ and $\vec{H}$, the conduction state ($\vec{v} = \vec{H} = \theta = 0$)

will be unstable if Re (p) > 0. For the onset of stationary instability p = 0 and therefore all time derivatives in (3.10) - (3.13) can be dropped.

Eliminating ζ, θ and h from (3.10) -(3.13) we obtain the equation

$$\Big[\nabla^6\Big(Q\frac{\partial^2}{\partial z^2}-\nabla^4\Big)-\nabla_1^2\Big(Q\frac{\partial^2}{\partial z^2}-\nabla^4\Big)R$$

$$-Q\frac{\partial^2}{\partial z^2}\nabla^2\Big(Q\frac{\partial^2}{\partial z^2}-\nabla^4\Big)-\tau^2\nabla^4\Big(\frac{\partial^2}{\partial z^2}\Big)\Big]w=0$$

(3.14)

3.3 Boundary Conditions

If the medium adjoining the fluid is assumed non-conducting, then

$$\vec{H}=\vec{H}^{ex}\quad,\quad J_z=0$$

where $\vec{H}^{ex}$ is the field appropriate to a vacuum such that

$\vec{H}^{ex}$ = grad ψ , $\nabla^2\psi = 0$ and on the boundary, the field is continuous.

However if we consider the medium adjoining the fluid as a perfect conductor then no magnetic field can cross the boundary and therefore h = 0

If rigid boundaries are considered then $J_x = J_y = 0$ as u and v are zero E_x and $E_y = E = 0$ on a boundary adjoining a perfect conductor

Using div $\vec{J} = 0$ and the fact that $J_x = J_y = 0$, we conclude that

$$\frac{\partial J_z}{\partial z} = 0$$

3.4 The onset of convection

In the z-direction, sin πz is the lowest mode that satisfies the idealized boundary conditions. Thus the operator $\frac{\partial^2}{\partial z^2}$ can be replaced by $-\pi^2$. Assuming the laterally infinite system, the convection pattern is periodic in x-y plane with wave number a

Therefore, ∇_1^2 can be replaced by $-a^2$ and ∇^2 by $-(\pi^2 + a^2)$.

From equation (3.14)

$$R(a) = \frac{(\pi^2+a^2)^3}{a^2} + \frac{Q\,\pi^2(\pi^2+a^2)}{a^2} + \frac{T\pi^2}{a^2} - \frac{Q\,T\,\pi^4}{a^2[(\pi^2+a^2)+Q\pi^2]} \qquad (3.15)$$

Thus we see that the stabilization by Q alone is expressed in the second term and that stabilization by T alone in the third term. Lowering of the Rayleigh number below the threshold is exhibited by the fourth term.

Minimizing R(a) with respect to a gives the critical wave number. The equation that `a_c' satisfies becomes

$$\frac{3(\pi^2+a_c^2)^2}{a_c^2} - \frac{(\pi^2+a_c^2)^3}{a_c^4} - \frac{Q\,\pi^4}{a_c^4} - \frac{T\,\pi^2}{a_c^4} + \frac{T\,Q\,\pi^4}{a_c^4}\left[2a_c^2(\pi^2+a_c^2) + Q\,\pi^2 + (\pi^2+a_c^2)^2\right]\left[(\pi^2+a_c^2)^2 + Q\,\pi^2\right]^{-2} = 0 \qquad (3.16)$$

3.5 Küppers-Lortz instability

For fluids with infinite Prandtl number, $\sigma_z \gg \sigma$. The time dependence is to be retained therefore only in equation (3.12).

In equation (3.13) if we operate $\frac{\partial}{\partial z}$ on both sides we obtain,

$$-\nabla^2 \frac{\partial h}{\partial z} = \frac{\partial^2}{\partial z^2} w \tag{3.17}$$

operating $\frac{\partial}{\partial z}$ on equation (3.11) gives

$$-\nabla^2 \frac{\partial \zeta}{\partial z} = \tau \frac{\partial^2}{\partial z^2} w + Q \frac{\partial^2}{\partial z^2} J \tag{3.18}$$

operating ∇^2 on equation (3.10)

$$\nabla^2 \left[\frac{\partial}{\partial t} \nabla^2 - \nabla^4 \right] w = R \nabla^2 \nabla_1^2 \theta - \tau \nabla^2 \frac{\partial \zeta}{\partial z} + Q \nabla^2 \left(\nabla^2 \frac{\partial h}{\partial z} \right) \tag{3.19}$$

Eliminating h by using (3.17), equation (3.19) reduces to

$$\nabla^2 \left[\frac{\partial}{\partial t} \nabla^2 - \nabla^4 \right] w = R \nabla^2 \nabla_1^2 \theta - \tau \nabla^2 \frac{\partial \zeta}{\partial z} - Q \nabla^2 \frac{\partial^2 w}{\partial z^2} \tag{3.20}$$

Consider ∇^2 of equation (3.18)

$$-\nabla^4 \zeta = \tau \frac{\partial}{\partial z} \nabla^2 w + Q \frac{\partial}{\partial z} \nabla^2 J \tag{3.21a}$$

Using equation (3.14) $-\nabla^2 J = \frac{\partial \zeta}{\partial z}$, equation (3.21) becomes

$$-\nabla^4 \zeta = \tau \frac{\partial}{\partial z} \nabla^2 w - Q \frac{\partial^2 \zeta}{\partial z^2} \tag{3.21b}$$

or, $$\left[\nabla^4 - Q \frac{\partial^2}{\partial z^2} \right] \zeta + \tau \frac{\partial}{\partial z} \nabla^2 w = 0 \tag{3.22}$$

$$(\nabla^2 - \sigma \frac{\partial}{\partial t}) \theta + w = (\vec{v} \cdot \vec{\nabla}) \theta \tag{3.23}$$

Thus the system has been reduced to three variables. Expressing w, ζ, θ in vectorial form,

$$X = \begin{bmatrix} w \\ \zeta \\ \theta \end{bmatrix}$$

a compact form for expressing (3.21) - (3.23) would be

$$L_o X + (\Delta R) L_1 X = N(X, X') + \sigma \frac{\partial}{\partial t} X_i \delta_{i3} \tag{3.24}$$

$N(X, X')$ is the column vector $\begin{bmatrix} 0 \\ 0 \\ (\vec{v} \cdot \vec{\nabla})\theta' \end{bmatrix}$

$\Delta R = R - R_c$ $\qquad \frac{\partial}{\partial z} = D$

L_o is the operator

$$\begin{bmatrix} \nabla^2(\nabla^4 - Q D^2) & -\tau \nabla^2 D & R_c \nabla_1^2 \nabla^2 \\ \tau D \nabla^2 & \nabla^4 - Q D^2 & 0 \\ 1 & 0 & \nabla^2 \end{bmatrix} \tag{3.25a}$$

and L_1 is

$$\begin{bmatrix} 0 & 0 & \nabla_1^2 \nabla^2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{3.25b}$$

Assuming that X represents the cylindrical rolls that are formed at the onset of convection, using perturbation theory by a power expansion in ΔR, it is possible to determine this state for slightly super - critical R .

Expressing the deviation in R from R_c as ΔR we expand

$$\Delta R = \sum_{n=1}^{\infty} \epsilon^n R_n \tag{3.26}$$

$$\vec{X} = \sum_{n=1}^{\infty} \epsilon^n \vec{X}_{n-1} \tag{3.27}$$

ϵ represents the amplitude of the non-linear convection . At the zeroth order in ϵ

$$\vec{X} = \begin{bmatrix} W_o \\ \zeta_o \\ \theta_o \end{bmatrix} , \qquad N(X',X) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{3.28}$$

$\Delta R = \epsilon R_1$

Equation (3.25) becomes

$$L_o \vec{X}_o = \sigma \frac{\partial}{\partial t} X_i \delta_{i3} \tag{3.29}$$

We obtain at the zeroth order

$$W_o = \cos a x \sin \pi z \tag{3.30a}$$

$$u_o = -\frac{\pi}{a} \sin a x \cos \pi z \tag{3.30b}$$

$$v_o = \frac{\pi \tau}{a} \frac{(\pi^2+a^2)}{(\pi^2+a^2)^2+\pi^2 Q} \sin a x \cos \pi z \tag{3.30c}$$

$$\zeta_o = \frac{\pi \tau (\pi^2+a^2)}{(\pi^2+a^2)^2+\pi^2 Q} \cos a x \cos \pi z \tag{3.30d}$$

$$\theta_o = \frac{1}{(\pi^2+a^2)} \sin \pi z \cos a x \tag{3.30e}$$

At the first order equation (3.25) becomes equivalent to

$$L_o \begin{bmatrix} w_1 \\ \zeta_1 \\ \theta_1 \end{bmatrix} + R_1 \begin{bmatrix} 0 & 0 & \nabla_1^2 \nabla^2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} w_o \\ \zeta_o \\ \theta_o \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ (\vec{v}_o \cdot \vec{\nabla})\theta_o \end{bmatrix} \quad (3.31)$$

Applying the solvability criteria described in Appendix B, we have

$$\langle X_o | N (X_o, Y_o) \rangle - R_1 \langle X_o | L_1 X_o \rangle = 0 \quad (3.32)$$

The first expression gives a zero contribution while $\langle X_o | L_1^{12} X_o \rangle$ is not equal to zero. This necessarily implies that $R_1 = 0$. We also have $w_1 = \zeta_1 = u_1 = v_1 = 0$ (3.33a)

$$\theta_1 = - \frac{\sin 2\pi z}{8 (\pi^2 + a^2)} \quad (3.33b)$$

At the next order , the equation becomes

$$L_o X_2 + R_1 L_1 X_1 + R_2 L_1 X_o = N (X_o, Y_1) + N (X_1, Y_o) \quad (3.34)$$

Using equation (3.34) ,we arrive at the solvability criteria as

$$- R_2 \langle X_o | L_1 X_o \rangle + \langle X_o | N (X_o, Y_1) \rangle = 0 \quad (3.35)$$

This results in $R_2 = \dfrac{R_o}{8 (\pi^2 + a^2)^2}$ (3.36)

Since $R_2 > 0$,it implies that the bifurcation to the convection state is positive for all values of τ and θ. Küppers and Loft have found the bifurcation to be greater than zero for Q = 0. This result is thus unaffected by the presence of the magnetic field.

The investigation of the stability of the cylindrical rolls against perturbations by another set is the basic feature to be explored for the study of Küppers Lortz instability. We assume a perturbation vector $\vec{Y}$ with time dependence of the form e^{pt}.A study of growth of this perturbation determines the stability of the

original set described by $\vec{X}$. With operators L_o and L_1 as described above , the hydrodynamic equation (3.21) - (3.23) can be represented as

$$- p \sigma M Y + L_o Y + \Delta R L_1 Y = N(X,Y) + N(Y,X) \tag{3.37}$$

where the matrix M is

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

We expand Y exactly as we represented X .Thus we have

$$Y = \sum_{n=1}^{\infty} \varepsilon^n Y_{n-1} \qquad p = \sum_{n=0}^{\infty} \varepsilon^n p_n \tag{3.38}$$

As we are considering the stability of the marginal state[7], at the zeroth order we choose $p_o = 0$ .If the perturbing roll is also a cylinder with the axis along the y direction ,the original roll remains stable.

At the zeroth order we have ,$L_o Y_o = 0$

$$\tilde{w}_o = \mathrm{Re} \left\{ e^{i (k_1 x + k_2 y)} \sin \pi z \right\} \tag{3.39a}$$

$$\tilde{u}_o = \mathrm{Re}\left[\left\{ \frac{i \pi \tau k_2 (\pi^2 + a^2)}{[(\pi^2 + a^2)^2 + \pi^2 Q] a^2} + \frac{i \pi k_1}{a^2} \right\} e^{i (k_1 x + k_2 y)} \right] \cos \pi z \tag{3.39b}$$

$$\tilde{v}_o = \mathrm{Re}\left[\left\{ - \frac{i\,\pi\,\tau\,k_1\,(\pi^2 + a^2)}{[\,(\pi^2 + a^2\,)^2 \;+ \pi^2\,Q\,]\,a^2} + \frac{i\,\pi\,k_2}{a^2} \right\} e^{\,i\,(k_1 x + k_2 y)} \right] \cos \pi z \tag{3.39c}$$

$$\tilde{\theta}_o = \mathrm{Re}\left[\frac{e^{i(\,k_1 x + k_2 y)}}{\pi^2 + a^2} \right] \sin \pi z \tag{3.39d}$$

$$\tilde{\zeta}_o = \mathrm{Re}\left[\frac{\pi\,\tau\,(\,\pi^2 + a^2)\,e^{\,i\,(k_1 x + k_2 y)}}{(\pi^2 + a^2)^2 + Q\,\pi^2} \right] \cos \pi z \tag{3.39e}$$

At the first order in ε , the equation reduces to

$$- p_1\,\sigma\;\; M\,Y_o \;+\; L_o\,Y_1 \;=\; N\,(X_o,Y_o) + N(Y_o,X_o) \tag{3.40}$$

Solvability condition becomes equivalent to

$$\langle\, Y_o |\; N\,(X_o,Y_o) + N\,(Y_o,X_o)\,\rangle + \langle\, Y_o |\; \sigma\,p_1 M\,Y_o \rangle = 0 \tag{3.41}$$

In equation (3.41), since the first term gives a zero contribution and since $\langle Y_o |\; M\,Y_o \rangle$ is not zero it necessarily implies that $p_1 = 0$.

Thus to investigate the Küppers Lortz instability, terms of higher order in ε need to be considered. With $p_1 = 0$, $R_1 = 0$ we obtain expressions for w_1, u_1, v_1, ζ_1 and θ_1 as

$$\tilde{w}_1 = \mathrm{Re}\left[\left\{ A_+ e^{i(k_1 + a)x} + A_- e^{i(k_1 - a)x} \right\} e^{ik_2 y} \right] \sin 2\pi z \qquad (3.42a)$$

$$\tilde{\zeta}_1 = \mathrm{Re}\left[\left\{ B_+ e^{i(k_1 + a)x} + B_- e^{i(k_1 - a)x} \right\} e^{ik_2 y} \right] \cos 2\pi z \qquad (3.42b)$$

$$\tilde{\theta}_1 = \mathrm{Re}\left[\left\{ C_+ e^{i(k_1 + a)x} + C_- e^{i(k_1 - a)x} \right\} e^{ik_2 y} \right] \sin 2\pi z \qquad (3.42c)$$

$$\tilde{u}_1 = \mathrm{Re}\left[\left\{ \left[\frac{(2\pi i\,(k_1 + a)\,A_+ + ik_2\,B_+)}{(k_1 + a)^2 + k_2^2} \right] e^{i(k_1 + a)x} + \left[\frac{(2\pi i\,(k_1 - a)\,A_- + ik_2\,B_-)}{(k_1 - a)^2 + k_2^2} \right] e^{i(k_1 - a)x} \right\} e^{ik_2 y} \cos 2\pi z \right] \qquad (3.42d)$$

$$\tilde{v}_1 = \mathrm{Re}\left[\left\{ \left(\frac{2\pi i\, k_2\, A_+ - i(k_1 + a)\, B_+}{(k_1 + a)^2 + k_2^2} \right) e^{i(k_1 + a)x} + \left(\frac{2\pi i\, k_2\, A_- - i(k_1 - a)\, B_-}{(k_1 - a)^2 + k_2^2} \right) e^{i(k_1 - a)x} \right\} e^{i k_2 y} \cos 2\pi z \right] \qquad (3.42e)$$

where

$$A_\pm = \frac{-2a^2 (1 \pm \cos\theta) R_c\, \pi (1 \mp \cos\theta)}{(\pi^2 + a^2)\ \Delta_\pm} \qquad (3.43a)$$

$$B_\pm = \frac{-2\pi^2 R_c \tau}{\pi^2 + a^2}\ \frac{(1\mp\cos\theta)[2a^2(1\pm\cos\theta)][2a^2(1\pm\cos\theta)+4\pi^2]}{[\{2a^2(1\pm\cos\theta)+4\pi^2\}^2+4\pi^2 Q]\ \Delta_\pm} \qquad (3.43b)$$

$$C_\pm = \frac{-\pi(1\mp\cos\theta)[\{2a^2(1\pm\cos\theta)+4\pi^2\}^3+4\pi^2 Q\{2a^2(1\pm\cos\theta)+4\pi^2\}]}{(\pi^2+a^2)[2a^2(1\pm\cos\theta)+4\pi^2]\ \Delta_\pm} \qquad (3.43c)$$

$$\frac{-\pi(1\mp\cos\theta)\ 4\pi^2 T\ [2a^2(1\pm\cos\theta)+4\pi^2]^2}{(\pi^2+a^2)[2a^2(1\pm\cos\theta)+4\pi^2]\ \Delta_\pm\ \ [\{2a^2(1\pm\cos\theta)+4\pi^2\}+4\pi^2 Q]}$$

and

$$\Delta_\pm = [2a^2(1\pm\cos\theta)+4\pi^2]^3 + 4\pi^2 Q[2a^2(1\pm\cos\theta)+4\pi^2] + \frac{4\pi^2[2a^2(1\pm\cos\theta)+4\pi^2]^2\ T}{[2a^2(1\pm\cos\theta)+4\pi^2]^2+4\pi^2 Q} - 2a^2 R_c (1\pm\cos\theta) \qquad (3.43d)$$

At the second order, we have from Eq. (3.37)

$$L_oY_2 = N(X_1,Y_o) + N(X_o,Y_1) + N(Y_o,X_1) + N(Y_1,X_o) - R_2L_1Y_o + p_2\sigma MY_o \tag{3.44}$$

and applying the solvability criterion arrive at

$$p_2 = \left\{ \frac{\sin\theta}{4(\pi^2+a^2)} \left[\frac{B_+}{2(1+\cos\theta)} - \frac{B_-}{2(1-\cos\theta)} \right] + \frac{\pi\tau\sin\theta\,(\pi^2+a^2)[C_+-C_-]}{4[(\Pi^2+a^2)^2+\pi^2Q]} + \frac{\pi}{4}\left[(1+\cos\theta)\,C_- + (1-\cos\theta)\,C_+\right] \right\} \tag{3.45}$$

For $\tau = 0$, p_2 is negative as it should be and increases as τ increases, having a zero, when

$$\frac{\sin\theta}{\pi^2+a^2}\left[\frac{B_+}{2(1+\cos\theta)} - \frac{B_-}{2(1-\cos\theta)}\right] + \pi\left[C_+(1-\cos\theta) + C_-(1+\cos\theta)\right] + \frac{\pi\tau\,(\pi^2+a^2)\sin\theta}{(\pi^2+a^2)^2+\pi^2Q}\left[C_+ - C_-\right] = 0 \tag{3.46}$$

The condition for Küppers-Lortz instability is thus given by Eq.(3.46).

3.6 Küppers Lortz instability with Galerkin truncation

We now study the Küppers Lortz instability using a Galerkin model. The minimal model is considered first for the calculation of the critical rotation speed. A more complete representation is then considered which though complicated,has the advantage of yielding realistic answers. For the stress free boundaries we consider the following representations for the rolls along the y axis

$$w = a(t)\cos ax\sin\pi z \tag{3.47a}$$

$$\zeta = f(t)\cos ax\cos\pi z \tag{3.47b}$$

$$\theta = b(t) \cos ax \sin \pi z + c(t) \sin 2\pi z \tag{3.47c}$$

$$h = h(t) \cos ax \cos \pi z \tag{3.47d}$$

$$j = j(t) \cos ax \sin \pi z \tag{3.47e}$$

Introduction of this model in the hydrodynamic equations results in a set of 6 ordinary differential equations corresponding to the time evolution of the coefficients of the Fourier modes. With X,Y,Z,G,H,J as the scaled version of the coefficients a(t),b(t),c(t),f(t),h(t),j(t) respectively, we have the Lorenz model as

$$\dot{X} = \sigma \left\{ - X + Y + tG - \frac{\pi^2 Q}{(\pi^2 + a^2)^2} H \right\} \tag{3.48a}$$

$$\dot{Y} = - Y + r X - X Z \tag{3.48b}$$

$$\dot{Z} = X Y - b Z \tag{3.48c}$$

$$\dot{G} = - \sigma \left\{ G + t X + \frac{\pi^2 Q J}{(\pi^2 + a^2)^2} \right\} \tag{3.48d}$$

$$\sigma_2 \dot{H} = X - H \tag{3.48e}$$

$$\sigma_2 \dot{J} = G - J \tag{3.48f}$$

The stationary state solution of these equations is represented by setting all time derivatives to be zero. This gives X = Y = Z = H = J = G = 0. This represents the state of no convection. However, this equilibrium state of the fluid is perturbed at some critical

temperature gradient which is determined by the linear stability analysis about the conduction state .As in the earlier cases , we superimpose x, y, z, g,h and j as the perturbations,on the conduction state and set up the linearized hydrodynamic equations in these perturbations. Assuming time dependence of the form of e^{pt} , the marginal state will be defined by Re (p) = 0. In general, the marginal state can be either stationary or oscillatory depending on whether Im (p) = 0 or not for Re (p) = 0. For the conducting fluid placed in a magnetic field the marginal state can be oscillatory only when the magnetic Prandtl number (σ_2 defined as $\frac{\nu}{\eta}$) is larger than the thermal Prandtl number ($\sigma = \frac{\nu}{\varkappa}$). This condition is equivalent to having $\frac{\varkappa}{\eta} > 1$. For terrestrial conditions , this is of the order of 10^{-3}. Hence we expect that the onset of instability is via a stationary state for fluids under consideration. Thus the critical Rayleigh number for the onset of convection is determined under the condition Im (p) = 0 for Re (p) = 0 .

The disturbances are thus temporarily guided by

$$\begin{vmatrix} -1 & 1 & t & -\frac{\pi^2 Q}{(\pi^2+a^2)^2} & 0 \\ r & -1 & 0 & 0 & 0 \\ -t & 0 & -1 & 0 & -\frac{\pi^2 Q}{(\pi^2+a^2)^2} \\ 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & -1 \end{vmatrix} = 0 \tag{3.49}$$

With the assumption of non zero values of the variables x,y,g,h and j. We notice that the equation for Z decouples when linearized in these perturbations. Equating the determinant of the above matrix to zero determines the critical Rayleigh number for the onset of stationary convection.

$$R = \frac{(\pi^2+a^2)^3}{a^2} + \frac{\pi^2 Q\,(\pi^2+a^2)}{a^2} + \frac{T\pi^2}{a^2} - \frac{Q\,T\,\pi^4}{a^2\left[\pi^2 Q + (\pi^2+a^2)^2\right]} \tag{3.50}$$

$$\text{with} \quad r = \frac{R\,a^2}{(\pi^2+a^2)^3} \tag{3.51}$$

The second and the third terms represent the inhibiting effect of the magnetic field and the rotation respectively. The last term expresses the reduction in the Rayleigh number when the fluid is subjected simultaneously to the two constraints. The critical wave number a_c is determined by minimizing R(a) with respect to a. Thus a_c satisfies the equation

$$\frac{3\,(\pi^2+a_c^2)^2}{a_c^2} - \frac{(\pi^2+a_c^2)^3}{a_c^4} - \frac{Q\,\pi^4}{a_c^4} - \frac{T\,\pi^2}{a_c^4} + \frac{T\,Q\,\pi^4\left[2\,a_c^2\,(\pi^2+a_c^2) + Q\,\pi^2 + (\pi^2+a_c^2)^2\right]}{a_c^4\,\left[(\pi^2+a_c^2)^2 + Q\,\pi^2\right]^2} = 0 \tag{3.52}$$

The critical Rayleigh number and the critical wave number determined are in complete agreement with the results arrived at

by considering perturbation techniques. Thus we conclude that our truncated system is an accurate model for the investigation of Küppers Lortz instability .

As the critical Rayleigh number depends only on the magnitude of $\vec{a}$,it is obvious that an infinite degeneracy exists. All steady state solutions which have the same wave number but different directions are equally preferred due to the isotropy in the horizontal directions. In order to consider which of the solutions will be physically realized ,we need to enter the non-linear regime. The steady state rolls that are formed at the onset are described as

$$Y = \left\{ 1 + \frac{t^2}{1 + \frac{\pi^2 Q}{(\pi^2 + a^2)^2}} + \frac{\pi^2 Q}{(\pi^2 + a^2)^2} \right\} X \qquad (3.53a)$$

$$G = \frac{-X t}{1 + \frac{\pi^2 Q}{(\pi^2 + a^2)^2}} \qquad (3.53b)$$

$$H = X \qquad (3.53c)$$

$$J = G \qquad (3.53d)$$

The steady state rolls assumed along the y axis which are described by equation (3.53a) - (3.53d) are then perturbed by rolls along an arbitrary direction. Thus we need to include the y dependence as well in the expansions for w, ζ, θ, h and j. We

therefore represent these variables as

$$w = a(t) \cos a x \sin \pi z + a_1(t) \cos (k_1 x + k_2 y) \sin \pi z \tag{3.54a}$$

$$\zeta = \cos \pi z \left\{ f(t) \cos a x + f_1(t) \cos (k_1 x + k_2 y) \right\} \tag{3.54b}$$

$$\begin{aligned} \theta &= \sin \pi z \left\{ b(t) \cos a x + b_1(t) \cos (k_1 x + k_2 y) \right\} \\ &+ \sin 2\pi z \left\{ c(t) + c_1(t) \cos (k_1 x + k_2 y + a x) + \right. \\ &\qquad \left. + c_2(t) \cos (k_1 x + k_2 y - a x) \right\} \end{aligned} \tag{3.54c}$$

$$h = \cos \pi z \left\{ h(t) \cos a x + h_1(t) \cos (k_1 x + k_2 y) \right\} \tag{3.54d}$$

$$j = \sin \pi z \left\{ j(t) \cos a x + j_1(t) \cos (k_1 x + k_2 y) \right\} \tag{3.54e}$$

Introduction of these expansions into the equations (3.21)- (3.23) results in a 13×13 Lorenz system.

The rolls along an arbitrary direction are described by the

$$Y_1 = \left\{ 1 + \frac{t^2}{1 + \dfrac{\pi^2 Q}{(\pi^2 + a^2)^2}} + \frac{\pi^2 Q}{(\pi^2 + a^2)^2} \right\} X_1 \tag{3.55a}$$

$$G_1 = \frac{- X_1 t}{1 + \dfrac{\pi^2 Q}{(\pi^2 + a^2)^2}} \tag{3.55b}$$

$$H_1 = X_1 \tag{3.55c}$$

$$J_1 = G_1 \tag{3.55d}$$

since these rolls satisfy exactly the same equations as the original rolls. The condition for Küppers Lortz instability for the minimal model is given by

$$c\ t \sin\theta = \left\{ 1 + \frac{\pi^2 Q}{(\pi^2 + a^2\ )^2} \right\} \frac{b_+(1+\cos\theta)^2+b_-(1-\cos\theta)^2}{b_+(1+\cos\theta) - b_-(1-\cos\theta)} \tag{3.56}$$

where b_+ and b_- are defined as $\quad b_\pm = \dfrac{4\pi^2 + 2a^2\ (1 \pm \cos\theta)}{\pi^2 + a^2}$ (3.57)

For $Q = 0$,this equation becomes identical to equation (2.26) From this expression we see that T is a monotonically increasing function of Q . The critical θ for the onset of Küppers Lortz instability is found to be about 58°. It is very weakly dependent on the magnetic field .

The minimal model that has been discussed above has the advantage of being handled easily . It is able to show all the qualitative features of Küppers Lortz instability in the presence of the magnetic field. However to obtain more accurate results we need to consider a higher order truncated model. We introduce additional terms in the w , ζ, h, j expansions. Accordingly we have

$$w = \sin \pi z \left\{ a(t) \cos a x + a_1(t) \cos (k_1 x + k_2 y) \right\} + \sin 2\pi z$$
$$\left\{ a_2(t) \cos (k_1 x + a x + k_2 y) + a_3(t) \cos (k_1 x - a x + k_2 y) \right\} \tag{3.58a}$$

$$\zeta = \cos \pi z \left\{ f(t) \cos a x + f_1(t) \cos (k_1 x + k_2 y) \right\} + \sin 2\pi z$$
$$\left\{ f_2(t) \cos (k_1 x + a x + k_2 y) + f_3(t) \cos (k_1 x - a x + k_2 y) \right\} \tag{3.58b}$$

$$\theta = \sin \pi z \left\{ b(t) \cos a x + b_1(t) \cos (k_1 x + k_2 y) \right\} + \sin 2\pi z$$
$$\left\{ c(t) + c_1(t) \cos(k_1 x + a x + k_2 y) + c_2(t) \cos(k_1 x - a x + k_2 y) \right\} \tag{3.58c}$$

$$h = \cos \pi z \left\{ h(t) \cos a x + h_1(t) \cos (k_1 x + k_2 y) \right\} + \cos 2\pi z$$
$$\left\{ h_2(t) \cos (k_1 x + k_2 y + a x) + h_3(t) \cos (k_1 x + k_2 y - a x) \right\} \tag{3.58d}$$

$$j = \sin \pi z \left\{ j(t) \cos a x + j_1(t) \cos (k_1 x + k_2 y) \right\} + \sin 2\pi z$$
$$\left\{ j_2(t) \cos (\quad + k_2 y + a x) + j_3(t) \cos (k_1 x - a x + k_2 y) \right\} \tag{3.58e}$$

Substitution of this model in the hydrodynamic equations results in a dynamical system of 21 equations.

$$\dot{X} = \sigma \left\{ -X + G t + Y - \frac{\pi^2 Q H}{(\pi^2 + a^2)^2} \right\} \tag{3.59a}$$

$$\dot{X}_1 = \sigma \left\{ -X_1 + G_1 t + Y_1 - \frac{\pi^2 Q H_1}{(\pi^2 + a^2)^2} \right\} \tag{3.59b}$$

$$\dot{X}_2 = \sigma \left\{ -b_+ X_2 + \frac{2 t G_2}{b_+} - \frac{2 \pi^2 Q H_2}{(\pi^2 + a^2)^2} \right\} \tag{3.59c}$$

$$\dot{X}_3 = \sigma \left\{ - b_- X_3 + \frac{2 t G_3}{b_-} - \frac{2 \pi^2 Q H_3}{(\pi^2 + a^2)^2} \right\} \tag{3.59d}$$

$$\dot{G} = \sigma \left\{ - G - tX - \frac{\pi^2 Q J}{(\pi^2 + a^2)^2} \right\} \tag{3.59e}$$

$$\dot{G}_1 = - \sigma \left\{ G_1 + tX_1 + \frac{\pi^2 Q J_1}{(\pi^2 + a^2)^2} \right\} \tag{3.59f}$$

$$\dot{G}_2 = - \sigma \left\{ 2 t X_2 + G_2 b_+ + \frac{2 \pi^2 Q J_2}{(\pi^2 + a^2)^2} \right\} \tag{3.59g}$$

$$\dot{G}_3 = - \sigma \left\{ 2 t X_3 + G_3 b_- + \frac{2 \pi^2 Q J_3}{(\pi^2 + a^2)^2} \right\} \tag{3.59h}$$

$$\dot{H} \sigma_2 = X - H \tag{3.59i}$$

$$\dot{H}_1 \sigma_2 = X_1 - H_1 \tag{3.59j}$$

$$\dot{H}_2 \sigma_2 = - b_+ H_2 + 2 X_2 \tag{3.59k}$$

$$\dot{H}_3 \sigma_2 = - b_- H_3 + 2 X_3 \tag{3.59l}$$

$$\dot{J}\,\sigma_2 = G - J \tag{3.59m}$$

$$\dot{J}_1\,\sigma_2 = G_1 - J_1 \tag{3.59n}$$

$$\dot{J}_3\,\sigma_2 = 2\,G_2 - b_-\,J_3 \tag{3.59p}$$

$$\dot{Y} = -X\,Z + r\,X - Y - \frac{Z_1}{4}\left\{X_1(\,1-\cos\theta\,) + cG_1\sin\theta\,)\right\} - \frac{Z_2}{4}\left\{X_1(\,1+\cos\theta\,) - c\,G_1\sin\theta\,)\right\} - Y_1\,c\,\sin\theta \left\{\frac{G_3}{8\,(\,1-\cos\theta\,)} - \frac{G_2}{8\,(\,1+\cos\theta\,)}\right\} \tag{3.59q}$$

$$\dot{Y}_1 = -X_1\,Z + r\,X_1 - Y_1 - \frac{Z_1}{4}\left\{(1-\cos\theta)X - c\,G\,\sin\theta\right\} - \frac{Z_2}{4}\left\{(1+\cos\theta)X + c\,G\,\sin\theta\right\} + Y\,c\,\sin\theta \left\{\frac{G_3}{8\,(\,1-\cos\theta\,)} - \frac{G_2}{8\,(\,1+\cos\theta\,)}\right\} \tag{3.59r}$$

$$\dot{Z} = X\,Y + X_1\,Y_1 - b\,Z \tag{3.59s}$$

$$\dot{Z}_1 = \frac{(\,1-\cos\theta\,)}{2}(X\,Y_1 + X_1\,Y) - b_+Z_1 - r\,X_2 + \frac{c\,\sin\theta}{2}(\,G_1\,Y - G\,Y_1\,) \tag{3.59t}$$

$$\dot{Z}_2 = \frac{(1 + \cos\theta)}{2} (X Y_1 + X_1 Y) - b_- Z_1 - r X_3 + \frac{c \sin\theta}{2} (G Y_1 - G_1 Y) \quad (3.59u)$$

We have rescaled time by a factor of σ.

Since we require to investigate the instability right at the threshold we drop all the terms containing the time derivatives. The condition for Küppers Lortz instability is then obtained by substituting the values of the variables from equation (3.59) after equating the time derivatives to zero in the expression for $\dot{Y}_1$ in equation (3.63g).

With $X_2 = -\dfrac{2 \sin^2\theta \; R a^2}{\Delta_+}$ and $X_3' = -\dfrac{2 \sin^2\theta \; R a^2}{\Delta_-}$

value of about 60°. The results that are obtained using the Galerkin truncation are in agreement with that obtained in section 3.5 using the perturbation techniques.

3.7 Küppers Lortz instability in finite Prandtl number fluids :

In this section, we lift the constraint of $\sigma \gg 1$. For fluids with low Prandtl number, the onset of convection is oscillatory when the fluid is rotated at a sufficiently high speed. However, instead of rotation, if we subject such a fluid to a magnetic field, the condition for the onset of convection as oscillatory is that the thermal Prandtl number has to be small in comparison to the magnetic Prandtl number. However under all terrestrial conditions, the ratio of (σ_2 / σ) is $O(10^{-5})$ and thus the onset of convection as stationary pattern is ensured. To investigate the Küppers-Lortz instability in fluids with low Prandtl number, we subject it simultaneously to rotation and magnetic field. The mode of instability at the onset is now a complicated function of the Taylor number, T, the Chandrashekhar number Q, thermal Prandtl number σ and the magnetic Prandtl number σ_2. We remain in that region of Q where the instability as stationary convection is ensured and in particular we investigate Küppers-Lortz instability in mercury.

The Hydrodynamic Equations

The non-linear terms in w and ζ can no longer be dropped since they are of $O(\sigma^{-1})$. As the non-linear terms in the equations for h and ζ are of $O(\sigma_2/\sigma)$ and in mercury (σ_2 / σ) is of the order of 10^{-5}, these terms can be neglected. The

non-linear term in the equation for θ being O(1) is retained.

The governing equations for the finite Prandtl number becomes

$$\nabla^2 \left[\frac{\partial}{\partial t} - \nabla^2 \right] w - \frac{1}{\sigma} \left[\vec{\nabla} \times \vec{\nabla} \times (\vec{v} \cdot \vec{\nabla}) \vec{v} \right]_z = R \nabla_1^2 \theta - \tau \frac{\partial \zeta}{\partial z}$$

$$+ Q \nabla^2 \frac{\partial h}{\partial z} \qquad (3.60a)$$

$$\left[\frac{\partial}{\partial t} - \nabla^2 \right] \zeta + \frac{1}{\sigma} \left[\vec{\nabla} \times (\vec{v} \cdot \vec{\nabla}) \vec{v} \right]_z = \tau \frac{\partial w}{\partial z} + Q \frac{\partial j}{\partial z} \qquad (3.60b)$$

$$\left[\sigma \frac{\partial}{\partial t} - \nabla^2 \right] \theta = w - (\vec{v} \cdot \vec{\nabla}) \theta \qquad (3.60c)$$

$$\left[\sigma_z \frac{\partial}{\partial t} - \nabla^2 \right] h = \frac{\partial w}{\partial z} \qquad (3.60d)$$

$$\left[\sigma_z \frac{\partial}{\partial t} - \nabla^2 \right] J = \frac{\partial \zeta}{\partial z} \qquad (3.60e)$$

The subscript z denotes the z-component of the additional non-linear terms.

We study the Küppers Lortz instability via the Galerkin truncated model used in Section 3.6. Introducing these representations in the governing equations result in the following system of equations.

$$\dot{X} = \sigma \left[- X + Y + t\,G - \frac{\pi^2 Q}{(\pi^2 + a^2)^2} H \right] - \frac{1}{4} \left[\frac{\pi^2}{(\pi^2 + a^2)} c X_1 \sin\theta \right.$$

$$\left\{ -G_2 \frac{(1-\cos\theta)}{(1+\cos\theta)} + G_3 \frac{(1+\cos\theta)}{(1-\cos\theta)} \right\} + G_1 \left\{ G_2(1-\cos\theta) + G_3(1+\cos\theta) \right\}$$

$$-X_1 X_2 (1-\cos\theta) - X_1 X_3 (1+\cos\theta) - c \sin\theta\, G_1 (X_2 - X_3)$$

$$\left. -X_1 \frac{\sin\theta}{2} \left\{ \frac{G_2}{(1+\cos\theta)} - \frac{G_3}{(1-\cos\theta)} \right\} \right] \quad (3.61a)$$

$$\dot{X}_1 = \sigma \left\{ -X_1 + Y_1 + t\,G_1 - \frac{\pi^2 Q}{(\pi^2 + a^2)^2} H_1 \right\} - \frac{1}{4} \left[\frac{\pi^2}{(\pi^2 + a^2)} c X \sin\theta \right.$$

$$\left\{ -G_2 \frac{(1-\cos\theta)}{(1+\cos\theta)} + G_3 \frac{(1+\cos\theta)}{(1-\cos\theta)} \right\} + G \left\{ G_2(1-\cos\theta) + G_3(1+\cos\theta) \right\}$$

$$-X X_2(1-\cos\theta) - X X_3(1+\cos\theta) + c \sin\theta\, G (X_2 - X_3)$$

$$\left. + X \frac{\sin\theta}{2} \left\{ \frac{G_2}{(1+\cos\theta)} - \frac{G_3}{(1-\cos\theta)} \right\} \right] \quad (3.61b)$$

$$\dot{X}_2 = \sigma \left\{ -b_+ X_2 + \frac{2\,t\,G_2}{b_+} - \frac{2\pi^2 Q}{(\pi^2 + a^2)^2} H_2 - \frac{2 z_1}{b_+} (1+\cos\theta) \right\}$$

$$- \frac{1}{4 b_+} \left[- \frac{8\pi^2}{(\pi^2 + a^2)} X X_1 (1-\cos\theta)^2 + \frac{8\pi^2}{(\pi^2 + a^2)} c \sin\theta (1-\cos\theta) \right.$$

$$(G X_1 - G_1 X) - 8 \sin^2\theta\, G G_1 + b_+ \left\{ 4 X X_1 (1-\cos\theta) \right.$$

$$\left. \left. + 2c \sin\theta (G_1 X - G X_1) \right\} \right] \quad (3.61c)$$

$$\dot{X}_3 = \sigma \left\{ -b_- X_3 + \frac{2\ t\ G_3}{b_-} - \frac{2\ \pi^2\ Q}{(\pi^2 + a^2)^2}\ H_3 - \frac{2\ z_2}{b_-}\ (1-\cos\theta) \right\}$$

$$- \frac{1}{4\ b_-} \left[- \frac{8\ \pi^2}{(\pi^2+a^2)}\ X\ X_1\ (1+\cos\theta)^2 - \frac{8\ \pi^2}{(\pi^2+a^2)}\ c\ \sin\theta\ (1+\cos\theta) \right.$$

$$(G\ X_1 - G_1\ X) - 8\ \sin^2\theta\ G\ G_1 + b_- \left\{ 4\ X\ X_1 (1-\cos\theta) \right.$$

$$\left. \left. - 2\ c\ \sin\theta\ (G_1\ X - G\ X_1) \right\} \right] \qquad (3.61d)$$

$$\dot{G} = \sigma \left\{ -G - t\ X - \frac{\pi^2\ Q\ J}{(\pi^2+a^2)^2} \right\} + \frac{1}{4} \left[X_1 G_2 (1-\cos\theta) + X_1 G_3 (1+\cos\theta) \right.$$

$$+ \frac{c\ \sin\theta}{2(1+\cos\theta)}\ G_1 G_2 (1+2\cos\theta) - \frac{c\ \sin\theta}{2(1-\cos\theta)}\ G_1 G_3 (1-2\cos\theta)$$

$$+ \frac{\pi^2}{(\pi^2+a^2)}\ c\ \sin\theta \left\{ X_1 X_2\ \frac{(1-\cos\theta)}{(1+\cos\theta)} - X_1 X_3\ \frac{(1+\cos\theta)}{(1-\cos\theta)} \right\}$$

$$\left. + G_1 \left\{ X_2 (1-\cos\theta) + X_3 (1+\cos\theta) \right\} \right] \qquad (3.61e)$$

$$\dot{G}_1 = \sigma \left\{ -G_1 - t\ X_1 - \frac{\pi^2\ Q}{(\pi^2+a^2)^2}\ J_1 \right\} + \frac{1}{4} \left[X\ G_2 (1-\cos\theta) \right.$$

$$+ X\ G_3 (1+\cos\theta) - \frac{c\ \sin\theta}{2(1+\cos\theta)} \left\{ G\ G_2 (1+2\cos\theta) \right\}$$

$$+ \frac{c\ \sin\theta}{2(1-\cos\theta)}\ G\ G_3 (1-2\cos\theta) - \frac{\pi^2}{(\pi^2+a^2)}\ c\ \sin\theta \left\{ X\ X_2\ \frac{(1-\cos\theta)}{(1+\cos\theta)} \right.$$

$$\left. \left. - X\ X_3\ \frac{(1+\cos\theta)}{(1-\cos\theta)} \right\} + G \left\{ X_2 (1-\cos\theta) + X_3 (1+\cos\theta) \right\} \right] \qquad (3.61f)$$

$$\dot{G}_2 = \sigma \left\{ -G_2\ \dot{b}_+ - 2\ t\ X_2 - \frac{2\ \pi^2\ Q}{(\pi^2+a^2)^2}\ J_2 \right\} \qquad (3.61g)$$

$$\dot{G}_3 = \sigma \left\{ -G_3\, b_- - 2\, t\, X_3 - \frac{2\, \pi^2\, Q}{(\pi^2 + a^2)^2}\, J_3 \right\} \tag{3.61h}$$

$$\sigma_2\, \dot{H}_1 = X_1 - H_1 \tag{3.61i}$$

$$\sigma_2\, \dot{H} = X - H \tag{3.61j}$$

$$\sigma_2\, \dot{H}_2 = 2\, X_2 - b_+\, H_2 \tag{3.61k}$$

$$\sigma_2\, \dot{H}_3 = 2\, X_3 - b_-\, H_3 \tag{3.61l}$$

$$\sigma_2\, \dot{J} = G - J \tag{3.61m}$$

$$\sigma_2\, \dot{J}_1 = G_1 - J_1 \tag{3.61n}$$

$$\sigma_2\, \dot{J}_2 = 2\, G_2 - J_2 \tag{3.61o}$$

$$\sigma_2\, \dot{J}_3 = 2\, G_3 - J_3 \tag{3.61p}$$

$$\dot{Y} = -X\, Z + r\, X - Y - X_1\, Z_1\, \frac{(1-\cos\theta)}{4} - X_1\, Z_2\, \frac{(1+\cos\theta)}{4}$$

$$- \frac{c \sin\theta}{4}\, G_1\, (Z_1 - Z_2) - \frac{Y_1 c \sin\theta}{4} \left\{ \frac{G_3}{2(1-\cos\theta)} - \frac{G_2}{2(1+\cos\theta)} \right\} \tag{3.61q}$$

$$\dot{Z} = X\, Y + X_1\, Y_1 - b\, Z \tag{3.61r}$$

$$\dot{Y}_1 = - X_1\, Z + r\, X_1 - Y_1 - \frac{X\, Z_1}{4}\, (1-\cos\theta) - \frac{X\, Z_2}{4}\, (1+\cos\theta)$$

$$+ \frac{c \sin\theta}{4}\, G\, (Z_1 - Z_2) + \frac{Y\, c \sin\theta}{4} \left\{ \frac{G_3}{2(1-\cos\theta)} - \frac{G_2}{2(1-\cos\theta)} \right\} \tag{3.61s}$$

$$\dot{Z}_1 = \frac{(1-\cos\theta)}{2}\,(\,Y\,X_1 + X\,Y_1) - b_+\,Z_1 + \frac{c\,\sin\theta}{2}\,(G_1 Y - G\,Y_1) - r\,X_2 \qquad (3.61t)$$

$$\dot{Z}_2 = \frac{(1+\cos\theta)}{2}\,(\,Y\,X_1 + X\,Y_1) - b_-\,Z_2 - \frac{c\,\sin\theta}{2}\,(G_1 Y - G\,Y_1) - r\,X_3 \qquad (3.61u)$$

In the above equations we have used

$$\left\{ \vec{\nabla} \times \vec{\nabla} \times (\,\vec{v}\circ\vec{\nabla}\,)\,\vec{v} \right\}_z = \frac{\partial}{\partial z}\left[\frac{\partial v_i}{\partial x_j}\;\frac{\partial v_j}{\partial x_i} \right\} - \nabla^2\,(\vec{v}\cdot\vec{\nabla}\,) \qquad (3.62a)$$

and

$$\left\{ \vec{\nabla} \times (\vec{v}\cdot\vec{\nabla}\,)\,\vec{v} \right\}_z = (\vec{v}\cdot\vec{\nabla}\,)\,\zeta - (\vec{\omega}\cdot\vec{\nabla}\,)\,w \qquad (3.62b)$$

Substituting for Y, Z, G in the expressions for Z_1 and Z_2 we arrive at

$$Z_1 = (1-\cos\theta)\,\frac{X\,X_1}{b_+}\left\{ 1 + \frac{t^2}{1+\pi^2 Q\,(\pi^2+a^2)^{-2}} + \frac{\pi^2\,Q}{(\pi^2+a^2)^2} \right\} - \frac{r\,X_2}{b_+} \qquad (3.63a)$$

$$Z_2 = (1+\cos\theta)\,\frac{X\,X_1}{b_-}\left\{ 1 + \frac{t^2}{1+\pi^2 Q\,(\pi^2+a^2)^{-2}} + \frac{\pi^2 Q}{(\pi^2+a^2)^2} \right\} - \frac{r\,X_3}{b_-} \qquad (3.63b)$$

From equations (3.61) - (3.63) we have

$$\dot{Y}_1 = - Z_1 \left\{ 1 - \cos\theta + \frac{c\ t \sin\theta}{1+\pi^2 Q\ (\pi^2+a^2)^{-2}} \right\} - Z_2 \left\{ 1+\cos\theta - \frac{c\ t \sin\theta}{1+\pi^2 Q\ (\pi^2+a^2)^{-2}} \right\} + ct \sin\theta \left\{ 1+\frac{\pi^2 Q}{(\pi^2+a^2)^2} + \frac{t^2}{1+\pi^2 Q(\pi^2+a^2)^{-2}} \right\}$$

$$\left[\frac{b_+ X_2}{b_+^2 + 4\pi^2 Q(\pi^2+a^2)^{-2}} \frac{1}{(1+\cos\theta)} - \frac{b_- X_2}{b_-^2 + 4\pi^2 Q(\pi^2+a^2)^{-2}} \frac{1}{(1-\cos\theta)} \right] \tag{3.64}$$

Where X_2 and X_3 are given by

$$X_2 = \frac{\dfrac{M}{b_+} + \dfrac{N_2}{\sigma}}{D_2}, \qquad X_3 = \frac{\dfrac{M}{b_-} + \dfrac{N_3}{\sigma}}{D_3}$$

$$M = -2 \sin^2\theta \left\{ 1 + \frac{\pi^2 Q}{(\pi^2 + a^2)^2} + \frac{t^2}{1 + \dfrac{\pi^2 Q}{(\pi^2+ a^2)^2}} \right\}$$

$$N_2 = \frac{2\pi^2 (1 - \cos\theta)^2}{(\pi^2+ a^2)} - b_+(1 - \cos\theta) - \frac{2 \sin^2\theta\ t^2}{\left[1 + \dfrac{\pi^2 Q}{(\pi^2+ a^2)^2}\right]^2}$$

$$D_2 = b_+^2 + \frac{4 t^2 b_+}{\dfrac{4\pi^2 Q}{(\pi^2+ a^2)^2} + b_+^2} + \frac{4\pi^2 Q}{(\pi^2+ a^2)^2} - \frac{2 r (1 + \cos\theta)}{b_+}$$

$$N_3 = \frac{2\,\pi^2\,(1+\cos\theta)^2}{(\pi^2+a^2)} - b_-(1+\cos\theta) - \frac{2\sin^2\theta\;t^2}{\left[\dfrac{\pi^2\,Q}{(\pi^2+a^2)^2}+1\right]^2}$$

$$D_3 = b_-^2 + \frac{4\,t^2\,b_-}{\dfrac{4\,\pi^2 Q}{(\pi^2+a^2)^2}+b_-^2} + \frac{4\,\pi^2\,Q}{(\pi^2+a^2)^2} - \frac{2\,r\,(1-\cos\theta)}{b_-}$$

The condition for the onset of Küppers Lortz instability is given by $\dot{Y}_1 = 0$. The effect of including the non linear terms in the equations for the velocity and vorticity is to reduce the critical Rayleigh number (fig.6). The critical angle is weakly dependent on the Prandtl number . Thus the presence of the magnetic field has enabled us to investigate Küppers Lortz instability in fluids with arbitrary Prandtl numbers .

CHAPTER IV

KÜPPERS LORTZ INSTABILITY IN FLUIDS WITH INSULATING BOUNDARIES

4.1 Introduction

We consider a horizontal layer of fluid rotating about a vertical axis to be confined between stress free boundaries. The thermal conditions on the boundaries are that of constant heat flux. We study the mode of instability that is preferred at the onset using linear stability analysis. We find that the onset of instability is in the form of long wavelength rolls,below a critical Taylor number. Next we address the question of Küppers Lortz instability in these long wavelength rolls. Investigation of Küppers Lortz instability is carried out via a Galerkin truncated model.

4.2 Hydrodynamic Equations in a dimensionless form

From Appendix A, the hydrodynamic equations in the dimensionless form are represented as

$$\nabla^2 \left(\nabla^2 - \frac{\partial}{\partial t} \right) w = \tau \frac{\partial \zeta}{\partial z} - R \nabla_1^2 \theta \tag{4.1a}$$

$$\left(\nabla^2 - \frac{\partial}{\partial t} \right) \zeta = - \tau \frac{\partial w}{\partial z} \tag{4.1b}$$

$$\left(\nabla^2 - \sigma \frac{\partial}{\partial t} \right) \theta = - w + (\vec{v} \cdot \vec{\nabla}) \theta \tag{4.1c}$$

4.3 Boundary Conditions

We consider the fluid to be confined within stress free boundaries and accordingly we have

$$w = \frac{\partial^2 w}{\partial z^2} = 0 \tag{4.2a}$$

$$\frac{\partial \zeta}{\partial z} = 0 \tag{4.2b}$$

The thermal conditions on the boundaries are that of constant heat flux, as we are assuming the fluid to be bounded by insulating surfaces. Thus unlike the case of perfectly conducting boundaries, where we had θ (the temperature fluctuations from the conduction state) equal to zero, we now have $\frac{\partial \theta}{\partial z} = 0$ (4.2c) on the boundaries.

4.4 The Lorenz model

A minimal truncated model which is consistent with the boundary conditions is assumed to be the solution of the equations (4.1a) - (4.1c). Introduction of the minimal model into these equations results in a system of 4 ordinary differential equations. Irrespective of the thermal conditions on the boundaries , the motion of the convecting fluid is described in terms of the variables w , ζ and θ .

$$w = a(t) \cos a x \sin \pi z \tag{4.3a}$$

$$\zeta = f(t) \cos a x \cos \pi z \tag{4.3b}$$

$$\theta = b(t) \cos a x + c(t) \cos \pi z \tag{4.3c}$$

Substitution of these representations in equations (4.1a)-

(4.1c) results in the dynamical system (after scaling time by σ)

$$\dot{X} = \sigma \left(- X + Y + t G \right) \tag{4.4a}$$

$$\dot{Y} = - X Z + \frac{8 r X}{\pi^2} - \frac{a^2 Y}{\pi^2 + a^2} \tag{4.4b}$$

$$\dot{Z} = X Y - \frac{\pi^2 Z}{\pi^2 + a^2} \tag{4.4c}$$

$$\dot{G} = - \sigma \left(t X + G \right) \tag{4.4d}$$

The equilibrium state of the system is defined by the solution of the equations (4.4a) -.(4.4b) under the condition of all time derivatives being zero. The steady state is thus defined by X,Y,Z and G are zero. This corresponds to a state of no convection. However as in the earlier cases the fluid remains stationary only below a critical temperature gradient characterized by the Rayleigh number.Above R_c, the conduction state gets destabilized in favour of a convection state. Whether the instability takes place via a stationary state or an oscillatory state is determined by the linear stability analysis. Employing this technique, we calculate the critical Rayleigh number for the onset of stationary as well as oscillatory instability. The marginal state would be the one corresponding to the lower Rayleigh number.In the limit of large σ , the onset is stationary while for σ tending towards zero the onset is oscillatory.

4.5 Linear Stability Analysis

We perturb the conduction state ($X = Y = Z = G = 0$) infinitesimally. These disturbances are assumed to have a time dependence of the form e^{pt}. We set up the hydrodynamic equations in these perturbations (neglecting all second and higher order terms in them) and determine the critical Rayleigh number and the critical wave number . If the onset is assumed to be stationary, we have $p = 0$ and the Rayleigh number determined under these conditions defines the threshold for stationary convection . If we assume the onset as oscillatory, we consider $p = \pm i\omega$ with ω as the frequency of oscillation and determine the critical Rayleigh number corresponding to it . As we are considering the marginal state the $\mathrm{Re}\,(p) = 0$.

We assume perturbations of the form x, y, z, g to be superimposed on the conduction state $X = Y = Z = G = 0$ and study the growth of the disturbance. Linearizing in the perturbations, the equation for z decouples and thus only the variables x, y, g need to be considered. These linearized hydrodynamic equations are

$$p\,x = \sigma\,(- x + y + t\,g) \tag{4.5a}$$

$$p\,g = -\sigma\,(t\,x + g) \tag{4.5b}$$

$$p\,y = \frac{8\,r\,x}{\pi^2} - \frac{a^2\,y}{\pi^2 + a^2} \tag{4.5c}$$

Equations (4.5a) - (4.5c) can be represented in a matrix form

as

$$\begin{bmatrix} p+\sigma & -\sigma & -\sigma t \\ -\dfrac{8r}{\pi^2} & p+a^2(\pi^2+a^2)^{-1} & 0 \\ \sigma t & 0 & p+\sigma \end{bmatrix} \begin{bmatrix} x \\ y \\ g \end{bmatrix} = 0$$

For a non trivial solution of these equations, the solvability condition is that the determinant of the matrix multiplying the variables x,y,g must necessarily be zero. This results in a cubic equation in p .

$$p^3 + p^2\left(2\sigma + \frac{a^2}{\pi^2+a^2}\right) + p\left(\sigma^2(1+t^2) - \frac{8r\sigma}{\pi^2} + \frac{2\sigma a^2}{\pi^2+a^2}\right) + \frac{\sigma^2 a^2(1+t^2)}{\pi^2+a^2} - \sigma^2\frac{8r}{\pi^2} = 0 \qquad (4.6)$$

For the onset to be stationary equation (4.6) is used for determining the critical Rayleigh number with the condition that p = 0. This is equivalent to having the constant term in equation (4.6) as zero. The critical Rayleigh number r_c is determined to be

$$r_c = \frac{\pi^2 a^2 (1+t^2)}{8(\pi^2+a^2)} \qquad (4.7)$$

Substituting $r_c = R_c \dfrac{a^2}{(\pi^2+a^2)^3}$

$$t^2 = \frac{\pi^2 T}{(\pi^2 + a^2)^3}$$

We obtain $$R_c(a) = \frac{\pi^2}{8}\left\{(\pi^2 + a^2)^2 + \frac{\pi^2 T}{\pi^2 + a^2}\right\} \qquad (4.8)$$

The critical wave number is determined by minimizing $R_c(a)$ with respect to a. Thus the critical wave number satisfies the equation

$$\pi^2 T = 2 (\pi^2 + a_c^2)^3 \qquad (4.9)$$

Thus introducing (4.9) in (4.8) determines the critical Rayleigh number.

$$R_c = \frac{3\pi^2}{8} (\pi^2 + a_c^2)^2 \qquad (4.10)$$

From equation (4.10) it is obvious that the minimum value of R_c will be obtained for $a_c = 0$. This corresponds to $R_c = 360$. In the absence of rotation, we have

$$R_c = \frac{\pi^2}{8} (\pi^2 + a_c^2)^2$$

The minimum is once again obtained for $a_c = 0$ corresponding to which the $R_c = 120$. This is in complete agreement with the results of Jakeman[17] for stress free surfaces. Thus in the presence of rotation,the critical Rayleigh number is raised by a factor of 3 if the fluid is confined within insulating boundaries .Corresponding to $a_c = 0$ we have ,

$R_1 (0) = \pi^2 (1 + T_1) / 8$

where $R_1 = \frac{R}{\pi^4}$ and $T_1 = \frac{T}{\pi^4}$

$$R_{1c}(0) = \frac{3\pi^2}{8}$$

Thus $R_1(0)$ will be a minimum if $R_1(0) < R_{1c}(0)$. This correponds to $T_1 \leq 2$. For $T_1 \leq 2$, we, therefore, have stationary convection in the form of long wavelength rolls. For $T_1 > 2$ stationary convection takes place in the form of finite wave length rolls. This threshold that we determine is independent of the Prandtl number.

Next we determine the critical Rayleigh number for the onset of instability as oscillatory convection. We substitute $p = i\omega$ which represents the marginal state and obtain the characteristic equation of the matrix.

$$-i\omega^3 - \omega^2\left(2\sigma + \frac{a^2}{\pi^2 + a^2}\right) + i\omega\left\{\sigma(1 + t^2) + \frac{2\sigma a^2}{\pi^2 + a^2} - \frac{8r\sigma}{\pi^2}\right\} + \frac{\sigma^2 a^2(1 + t^2)}{\pi^2 + a^2} - \frac{\sigma^2 8r}{\pi^2} = 0 \quad (4.11)$$

Equating the real and imaginary parts to zero we obtain

$$r_{os} = \left\{ \frac{\frac{2\sigma a^2}{\pi^2 + a^2} + \sigma^2(1 + t^2) + \frac{a^4}{(\pi^2 + a^2)^2}}{\sigma + \frac{a^2}{\pi^2 + a^2}} \right\} \frac{\pi^2}{4} \quad (4.12)$$

where $r_{os} = R_{os} \frac{a^2}{(\pi^2 + a^2)^3}$ with R_{os} representing the

critical Rayleigh number for the onset to be oscillatory. The frequency

$$\omega^2 = \sigma^2 (1+t^2) + \frac{2 \sigma a^2}{\pi^2 + a^2} - \frac{8 r \sigma}{\pi^2} \qquad (4.13)$$

4.6 Küppers Lortz instability

The stability of the long wave length rolls is determined by investigating the Küppers Lortz instability of such rolls. Accordingly, we superimpose perturbations in the form of a set of rolls with the axis along an arbitrary direction. Thus we represent the set of rolls along an arbitrary direction with modes of the form $\cos (k_1 x + k_2 y)$ for the x and y dependence.Here k_1 and k_2 are the components of $\vec{a}$ along the x and y directions respectively.

A minimal model representation for w, ζ and θ consistent with the conditions on the boundaries for being stress free and thermally insulating is

$$w = \sin \pi z \left\{ a(t) \cos ax + a_1(t) \cos (k_1 x + k_2 y) \right\} \qquad (4.14a)$$

$$\zeta = \cos \pi z \left\{ f(t) \cos ax + f_1(t) \cos(k_1 x + k_2 y) \right\} \qquad (4.14b)$$

$$\begin{aligned} \theta = {} & b(t) \cos ax + c(t) \cos \pi z + b_1(t) \cos (k_1 x + k_2 y) \\ & + \cos \pi z \{ c_1(t) \cos (k_1 x + k_2 y + a x)\} \\ & + \cos \pi z \{ c_2(t) \cos (k_1 x + k_2 y - a x)\} \end{aligned} \qquad (4.14c)$$

Introducing these representations in equations (4.1a)-(4.1c), we obtain the following dynamical system, with the time rescaled

$$\dot{X} = \sigma (- X + Y + t G) \tag{4.15a}$$

$$\dot{Y} = - \frac{a^2}{(\pi^2 + a^2)} Y + \frac{8 r X}{\pi^2} + \frac{\cos \theta}{2} (X_1 Z_1 - X_1 Z_2)$$

$$+ \frac{c \sin \theta}{2} (G_1 Z_2 - G_1 Z_1) - X Z \tag{4.15b}$$

$$\dot{Z} = - \frac{\pi^2 Z}{(\pi^2 + a^2)} + X Y + X_1 Y_1 \tag{4.15c}$$

$$\dot{G} = - \sigma (G + t X) \tag{4.15d}$$

$$\dot{X}_1 = \sigma (- X_1 + Y_1 + t G_1) \tag{4.15e}$$

$$\dot{Y}_1 = - \frac{a^2}{(\pi^2 + a^2)} Y_1 + \frac{8 r X_1}{\pi^2} + \frac{\cos \theta}{2} (X Z_1 - X Z_2)$$

$$+ \frac{c \sin \theta}{2} G (Z_1 - Z_2) - X_1 Z \tag{4.15f}$$

$$\dot{Z}_1 = - b_+ Z_1 - \cos \theta (X_1 Y + X Y_1) + c \sin \theta (Y G_1 - Y_1 G) \tag{4.15g}$$

$$\dot{Z}_2 = - b_- Z_2 + \cos \theta (X_1 Y + X Y_1) + c \sin \theta (Y_1 G - Y G_1) \tag{4.15h}$$

The rolls along the y axis are described by X,Y,Z,G non zero and the others zero. We consider the other set of rolls as the perturbation . A study of time evolution of these perturbations

indicate that the condition for the onset of Küppers Lortz instability is given as

$$(c\, t \sin\theta - \cos\theta)\,(Z_1 - Z_2) = 0 \qquad (4.16)$$

$c\, t \sin\theta = \cos\theta$ is the condition for Küppers Lortz instability .The minimum value of t thus obtained is zero and it corresponds to a critical angle of 90°.For $t > 0$, the basic pattern is destabilized by those rolls which are at an angle for which (fig. 7) $c\, t \geq \cot\theta$. Finally for very high values of t the original roll is unstable to a large range of rolls.

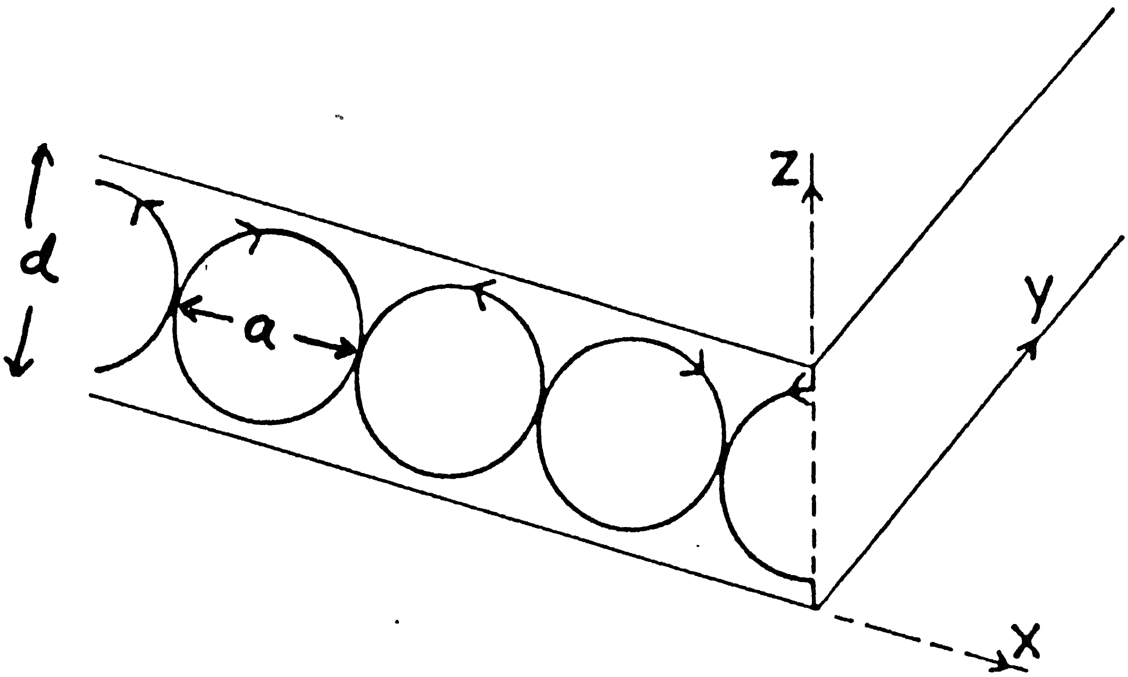


Fig 1 Convection rolls in RB geometry

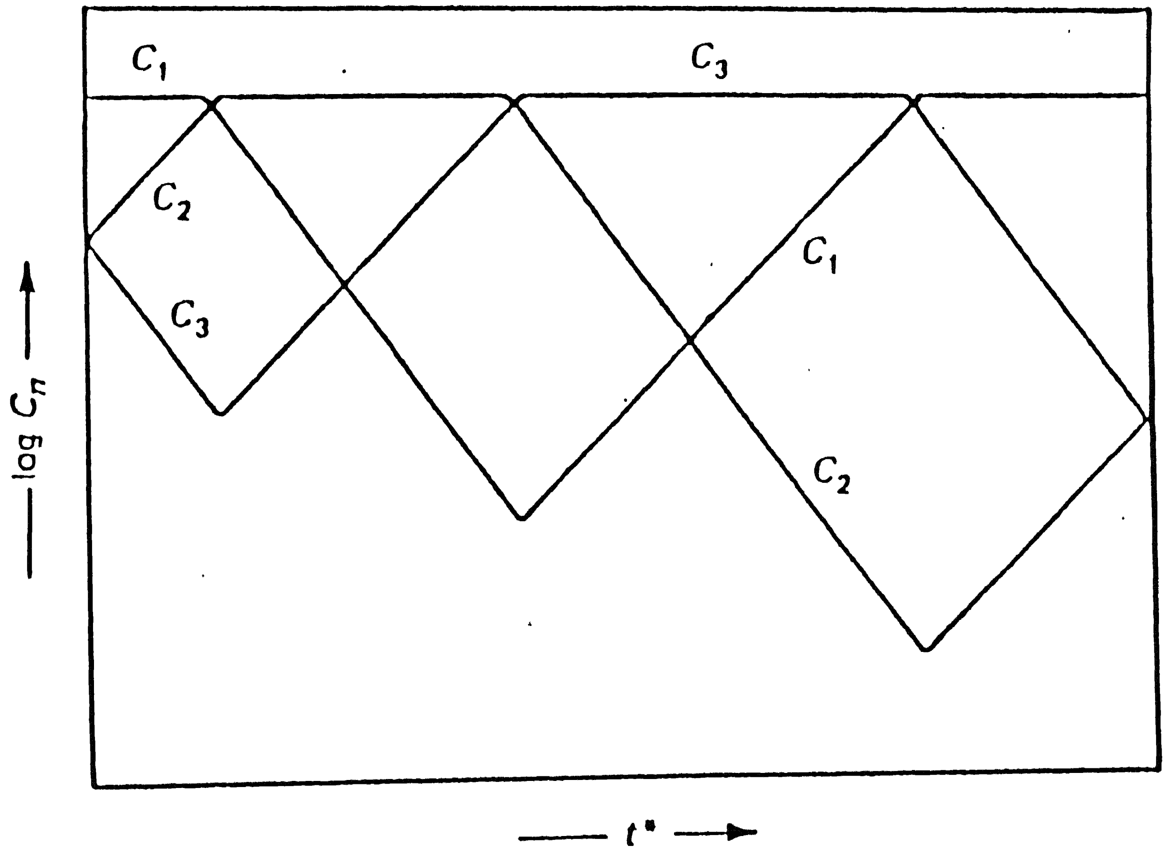
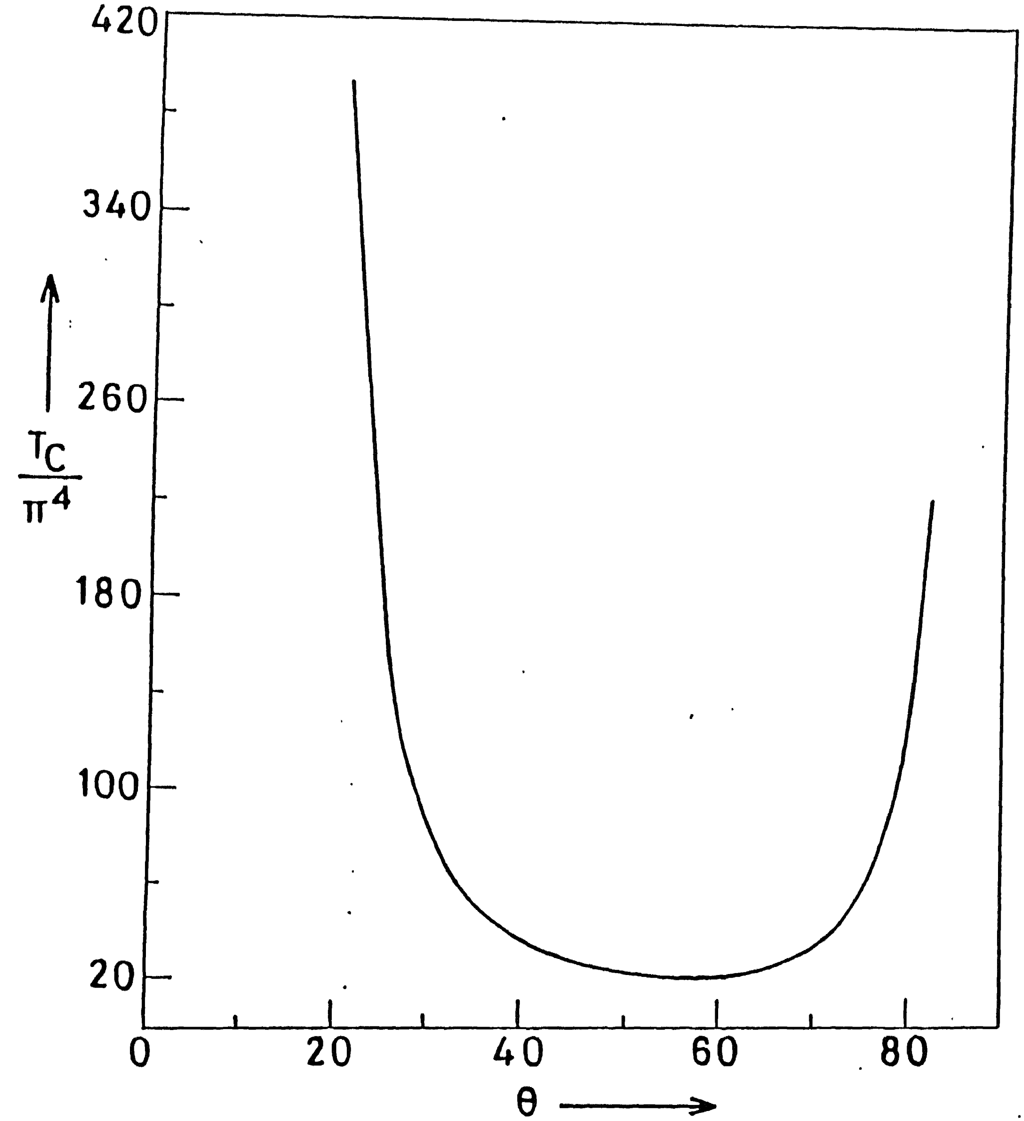


Fig.2 Relative growth of rolls with time



ig 3 The Critical Taylor Number T_c as a Function of angle θ

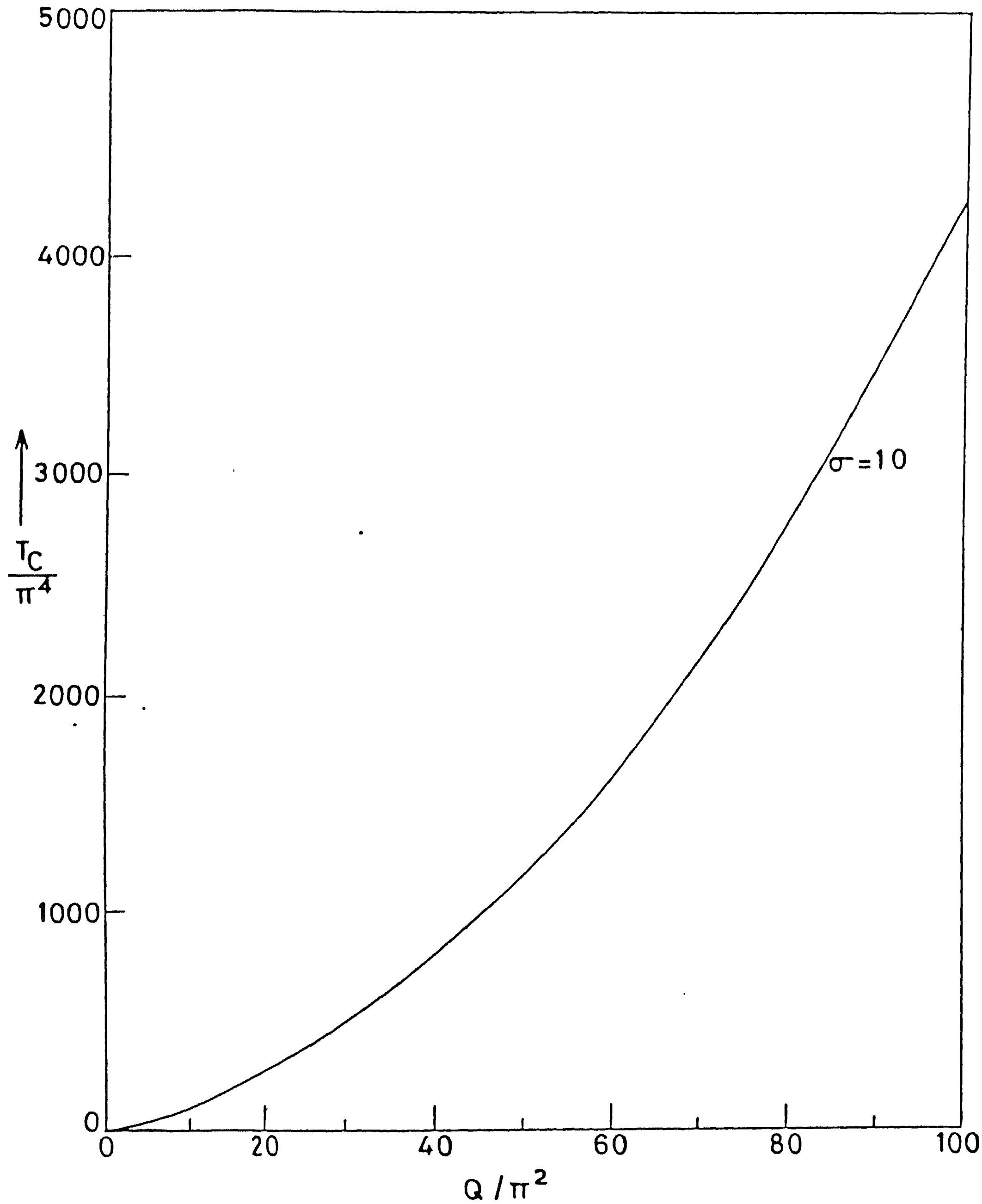


Fig 4 The critical Taylor number T_c as a function of the Chandrasekhar number Q.

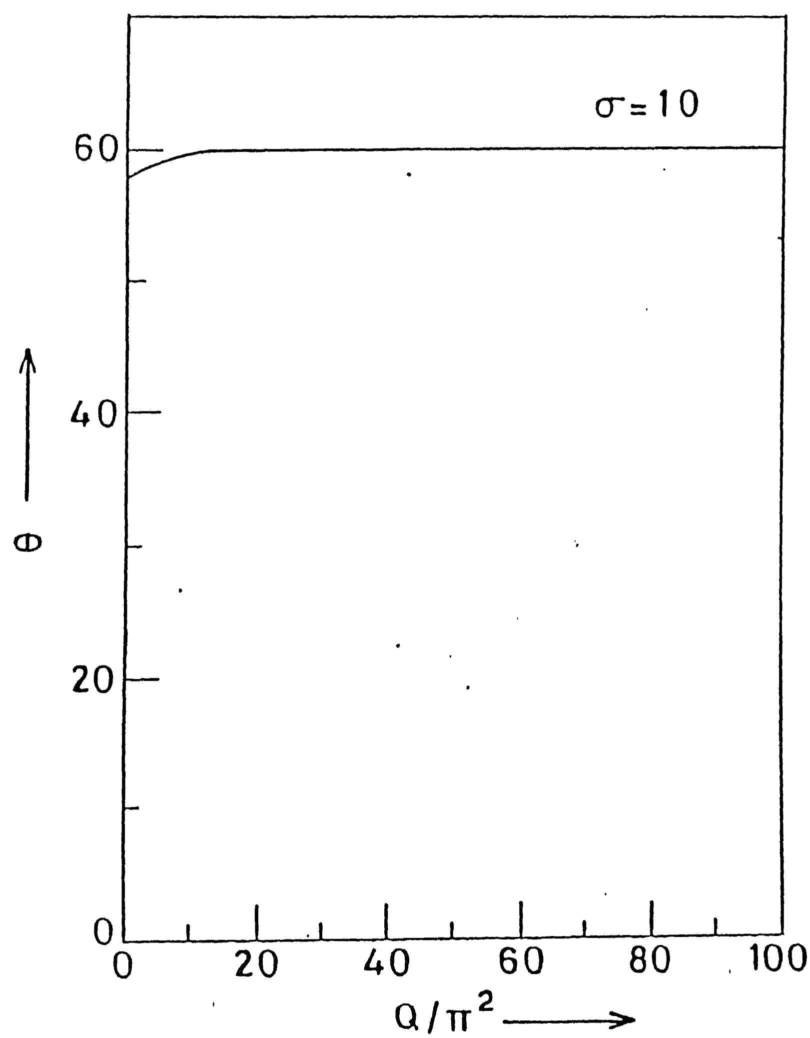


Fig 5 The critical angle θ as a function of the Chandrasekhar number Q.

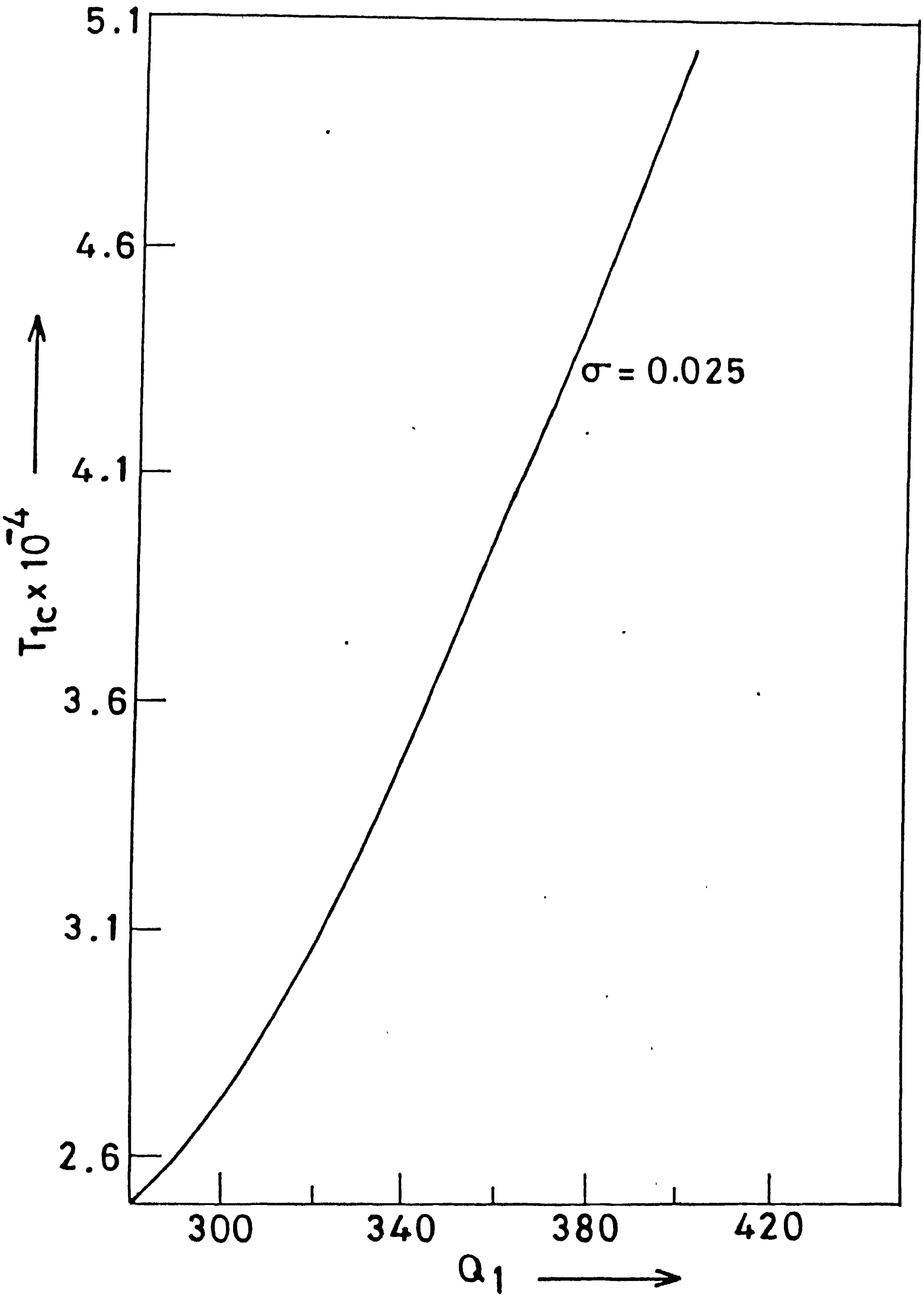


Fig 6 The Critical Taylor number $T1_c$ as a function of the Chandrasekhar number Q1 for finite Prandtl number.

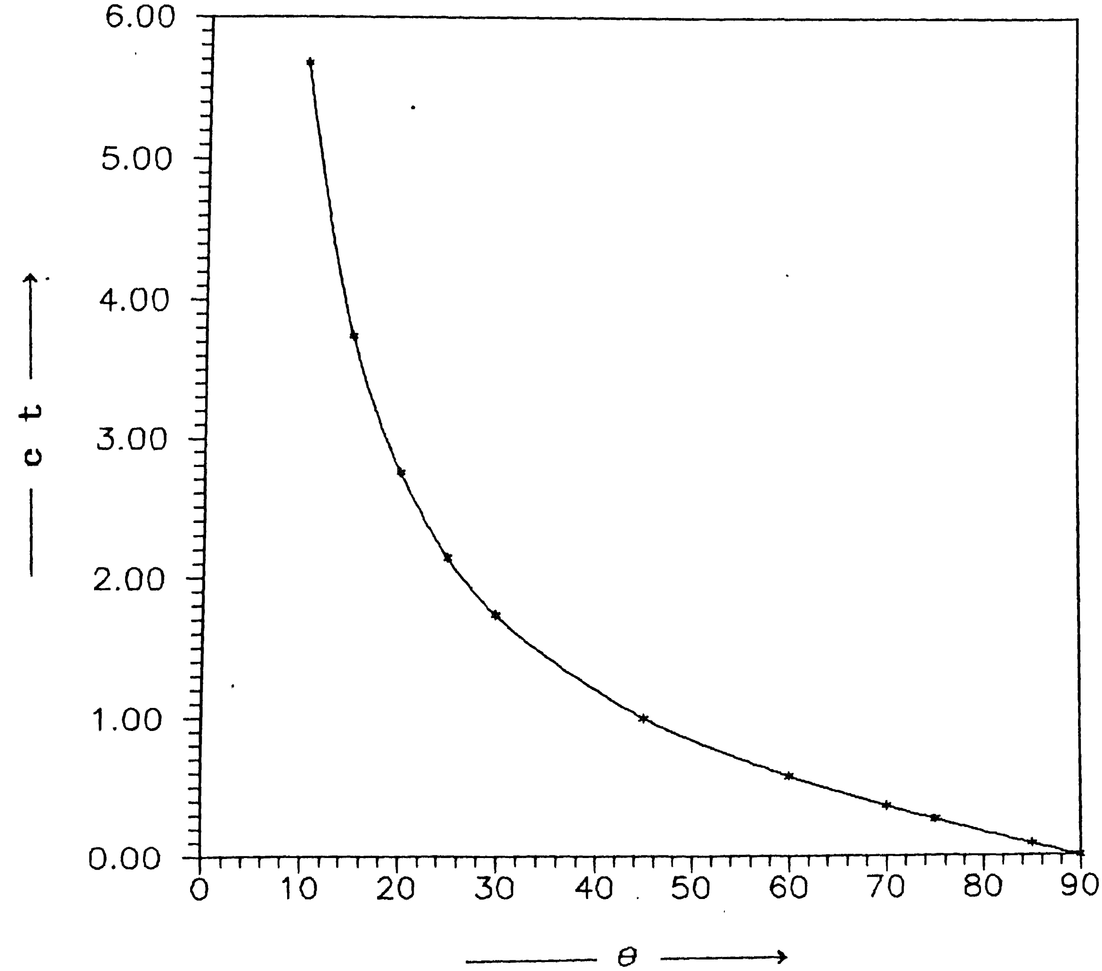


Fig. 7 The critical Taylor number against the angle
(For insulating boundaries).

APPENDIX A

In this appendix, we reduce the hydrodynamic equations to a scalar form.The equation of motion for the convecting fluid in the presence of rotation is given by equations (2.5a)-(2.5b).

Taking the curl on both sides of equation (2.5a) we get

$$\frac{\partial \vec{\omega}}{\partial t} + \{\vec{\nabla} \times (\vec{v}.\vec{\nabla})\vec{v}\} = \vec{\nabla} \times (\vec{g}\, \alpha\, \theta) + \nu \nabla^{2}\vec{\omega} + 2\, \Omega \frac{\partial \vec{v}}{\partial z} \tag{A.1}$$

Where $\vec{\omega}$ is the vorticity

We consider the z component of this equation and arrive at the expression for ζ (2.16b).

$$(\nabla^{2} - \frac{\partial}{\partial t}) \zeta = - \tau \frac{\partial w}{\partial z}$$

Considering the curl of A.1 once again and taking its z-component results in the expression for w which is given by equation (2.16a).

Thus the equations has been transformed into a scalar form which are simpler to handle.

APPENDIX B

i). Solvability criterion

We consider an inhomogeneous equation

$$L \mid f \rangle = \mid h \rangle \tag{B1}$$

which is to be solved under the constraint

$$L \mid g \rangle = 0 \tag{B2}$$

A vector $\langle g_1 |$ can be constructed with the property

$$\langle g_1 \mid L = 0 \tag{B3}$$

Then from equation (B1) we have the solvability condition

$$\langle g_1 \mid L \mid f \rangle = \langle g_1 \mid h \rangle = 0 \tag{B4}$$

Thus the condition is equivalent to stating that the inhomogeneous equations can be solved if the inhomogeneity is orthogonal to the solutions of the adjoint homogeneous problem.

ii) Newton Raphson method

We employ the Newton Raphson method for determining the critical T for the onset of Küppers Lortz instability. This is an iteration method for solving the equations $f(x) = 0$. A series is generated which gives a sequence x_o, x_1 ..x_n recursively using the relation

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \tag{B5}$$

where $f'(x_n)$ denotes the derivative of $f(x)$ at $x = x_n$. An initial guess x_o however is required to get the iteration started.

APPENDIX C

Integration of ordinary differential equations

We employ the Runge- Kutta method of order four for the numerical integration of the dynamical system which in general can be represented as

$$\frac{dy}{dx} = f(x,y) \tag{C1}$$

The formulae for the step by step integration which finally leads to a time series are

$$y_{n+1} = y_n + \frac{1}{6}\left(k_1 + 2\,k_2 + 2\,k_3 + k_4 \right) \tag{C2}$$

$$x_{n+1} = x_n + h$$

where h is the time interval per step.

$$k_1 = h_n\, f\,(x_n,\, y_n) \tag{C4}$$

$$k_2 = h_n\, f\,(x_n + 0.5\, h_n\,,\, y_n + 0.5\, k_1) \tag{C5}$$

$$k_3 = h_n\, f\,(x_n + 0.5\, h_n\,,\, y_n + 0.5\, k_2) \tag{C6}$$

$$k_4 = h_n\, f\,(x_n + h_n\,,\, y_n + k_3\,) \tag{C7}$$

the truncation error is estimated to be of the order of h^4.

APPENDIX - D

Küppers Lortz instability with rigid boundaries

In this appendix, we study the effect of rigid boundaries on the onset of Küppers Lortz instability. The boundary conditions to be satisfied by the velocity and vorticity modes are

$$\left.\begin{aligned} w &= \frac{\partial w}{\partial z} = 0 \\ \zeta &= 0 \end{aligned}\right\} \text{ at } z = \pm \frac{1}{2} \qquad \text{(D1)}$$

While the assumption of the boundaries being perfect thermal conductors leads to

$$\theta = 0 \qquad \text{at } z = \pm \frac{1}{2}$$

The rolls along the y-axis are represented as

$$w = a(t) \cos a x \left(z^2 - \frac{1}{4} \right)^2 \qquad \text{(D2)}$$

$$\zeta = f(t) \cos a x \, z \left(z^2 - \frac{1}{4} \right) \qquad \text{(D3)}$$

$$\theta = b(t) \cos a x \left(z^2 - \frac{1}{4} \right) + c(t) \, z \left(z^2 - \frac{1}{4} \right) \qquad \text{(D4)}$$

We substitute these expansions in the hydrodynamic equations and obtain a set of Lorenz equations which are given by

$$\dot{X} = \sigma (- X + Y + t G) \qquad \text{(D5)}$$

$$p_2 \dot{Y} = - Y + r X - \frac{X Z}{28} \qquad \text{(D6)}$$

$$p_3 \dot{Z} = X Y - b Z \qquad \text{(D7)}$$

$$p_1 \dot{G} = -\sigma \{ X t + (42 + a^2) G \} \tag{D8}$$

where $$r = \frac{189}{196} \frac{R a^2}{(10 + a^2)} (a^2 + 24 a^2 + 504) \tag{D9}$$

$$p_1 = \frac{(a^4 + 24 a^2 + 504)}{(12 + a^2)} , \qquad p_2 = \frac{p_1}{(10 + a^2)}$$

$$p_3 = p_1 (10 + a^2) , \qquad b = 42 (10 + a^2)$$

$$t^2 = \frac{12 \dot{\tau}^2}{(a^4 + 24 a^2 + 504)}$$

Performing the linear stability analysis about the conduction state (X = Y = Z = G = 0), we determine the critical Rayleigh number for the onset of convection as

$$r = 1 + \frac{t^2}{(42 + a^2)}$$

Thus we find that the threshold Rayleigh number is reduced by the introduction of rigid boundaries { $r = (1 + t^2)$ for free boundaries }. Minimizing with respect to `a' results in the determination of the critical wave number.

$$\frac{(42 + a^2)^2 (2 a^6 + 34 a^4 - 5040)}{(a^4 + 20 a^2 + 420)} = 12 \tau^2 \tag{D10}$$

In order to study the Küppers Lortz instability, of these rolls, we need to introduce a y dependence in the velocity, vorticity and temperature modes. Assuming

$$k_1 = a \cos \theta , \qquad k_2 = a \sin \theta$$

we now superimpose a set of rolls whose axis is along an arbitrary direction on the original set. Considering modes upto

second order we have

$$w = \left(z^2 - \frac{1}{4} \right)^2 \left[a(t) \cos a x + a_1(t) \cos (k_1 x + k_2 y) + \right.$$

$$\left. z\, a_2(t) \cos (k_1 x + k_2 y + a x) + z\, a_3(t) \cos (k_1 x + k_2 y - a x) \right]$$

(D11)

$$\zeta = \left(z^2 - \frac{1}{4} \right) \left[z\, f(t) \cos a x + z\, f_1(t) \cos (k_1 x + k_2 y) + \right.$$

$$\left. f_2(t) \cos (k_1 x + k_2 y + a x) + f_3(t) \cos (k_1 x + k_2 y - a x) \right]$$

(D12)

$$\theta = \left[\left(z^2 - \frac{1}{4} \right) \left\{ b(t) \cos a x + b_1(t) \cos (k_1 x + k_2 y) \right\} + \right.$$

$$z \left(z^2 - \frac{1}{4} \right) \left\{ c(t) + c_1(t) \cos (k_1 x + k_2 y + a x) + \right.$$

$$\left. \left. c_2(t) \cos (k_1 x + k_2 y - a x) \right\} \right] \qquad \text{(D13)}$$

Substitution of these expansions in the hydrodynamic equations results in

$$\dot{X}_2 \left\{ \frac{44 + 2 a^2 (1 + \cos \theta)}{(12 + a^2)} \right\} = \left\{ - t\, G_2 + Z_1 (1 + \cos \theta) - \frac{M+}{N} X_2 \right. \qquad \text{(D14a)}$$

$$\dot{X}_3 \left\{ \frac{44 + 2 a^2 (1 - \cos \theta)}{(12 + a^2)} \right\} = - t\, G + z_2 (1 - \cos \theta) - \frac{M-}{N} X_3 \qquad \text{(D14b)}$$

$$p_1 \dot{G}_2 = - \sigma \left[t X_2 + G_2 \left\{ 10 + 2 a^2 (1 + \cos \theta) \right\} \right] \qquad \text{(D14c)}$$

$$p_1 \dot{G}_3 = - \sigma \left[t X_3 + G_3 \left\{ 10 + 2 a^2 (1 - \cos \theta) \right\} \right] \qquad \text{(D14d)}$$

$$p_2 \dot{Y}_1 = r X_1 - Y_1 - \frac{X_1 Z}{28} - \frac{X Z_1 (1-2 \cos \theta)}{168} - \frac{X Z_2 (1+2 \cos\theta)}{168}$$

$$+ \frac{c G \sin \theta}{168} (z_1 - z_2) + c Y + \frac{c Y \sin \theta}{84} \left\{ \frac{G_3}{(1-\cos \theta)} - \frac{G_2}{(1+\cos \theta)} \right\}$$

(D14e)

$$p_2 \dot{Z}_2 = - b_- z_2 + \frac{(1+2 \cos \theta)}{3} (X Y_1 + X_1 Y) + \frac{c \sin \theta}{3} (G Y_1 - Y G_1)$$

$$+ r X_3 \frac{154}{81} (10+a^2) \qquad \text{(D14f)}$$

$$p_2 \dot{Z}_1 = - b_+ Z_1 + \frac{(1-2 \cos \theta)}{3} (X Y_1 + X_1 Y) + \frac{c \sin \theta}{3} (G Y_1 - Y G_1)$$

$$+ r X_2 \frac{154}{81} (10+a^2) \qquad \text{(D14g)}$$

$$\dot{X} = \sigma (- X + Y + t G) \qquad \text{(D14h)}$$

$$p_1 \dot{G} = - \sigma \left\{ t X + G (42 + a^2) \right\} \qquad \text{(D14i)}$$

$$p_2 \dot{Y} = r X - \frac{X Z}{28} - Y - X_1 Z_1 \frac{(1- 2 \cos \theta)}{168} - X_1 Z_2 \frac{(1-2 \cos\theta)}{168}$$

$$- \frac{G_1 c \sin \theta}{168} (Z_1 - Z_2) - \frac{Y c \sin \theta}{84} \left\{ \frac{G_3}{(1- \cos \theta)} - \frac{G_2}{(1 + \cos \theta)} \right\}$$

(D14j)

$$p_3 \dot{Z} = X_1 Y_1 + X Y - b Z \tag{D14k}$$

$$\dot{X}_1 = \sigma (- X_1 + Y_1 + t G_1) \tag{D14l}$$

$$p_1 \dot{G}_1 = - \sigma \left\{ t X_1 + (42 + a^2) G_1 \right\} \tag{D14m}$$

where $M_{\pm} = 3960 + \left\{ 2 a^2 (1 \pm \cos \theta) \right\}^2 + 176 a^2 (1 \pm \cos \theta)$

$$N = \left[a^4 + 24 a^2 + 504 \right] , \quad c = \left[\frac{a^4 + 24 a^2 + 504}{3} \right]^{\frac{1}{2}}$$

$$b_{\pm} = \left\{ 42 + 2 a^2 (1 \pm \cos \theta) \right\} \left\{ 10 + a^2 \right\}$$

The condition for Küppers Lortz instability is obtained as

$$- \frac{X Z_1}{168} \left\{ 1 - 2 \cos \theta + \frac{c t \sin \theta}{(42 + a^2)} \right\} - \frac{X Z_2}{168} \left\{ 1 + 2 \cos \theta - \frac{c t \sin \theta}{(42 + a^2)} \right\} + \frac{Y c \sin \theta}{84} \left\{ \frac{G_3}{(1 - \cos \theta)} - \frac{G_2}{(1 + \cos \theta)} \right\}$$

Using equations (D14a)- (D14m) we obtain the critical Taylor number to be of the order of $23 \pi^4$. In the limit of large Prandtl number, thus the boundary conditions have almost negligible effect on the onset of Küppers Lortz instability. The boundary conditions probably play a more important role in the limit of small Prandtl number.

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